

DeWitt-Schwinger Renormalization and Vacuum Polarization in d Dimensions

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Abstract

Calculation of the vacuum polarization, $\langle\phi^2(x)\rangle$, and expectation value of the stress tensor, $\langle T_{\mu\nu}(x)\rangle$, has seen a recent resurgence, notably for black hole spacetimes. To date, most calculations of this type have been done only in four dimensions. Extending these calculations to d dimensions includes d -dimensional renormalization. Typically, the renormalizing terms are found from Christensen's covariant point splitting method for the DeWitt-Schwinger expansion. However, some manipulation is required to put the correct terms into a form that is compatible with problems of the vacuum polarization type. Here, after a review of the current state of affairs for $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ calculations and a thorough introduction to the method of calculating $\langle\phi^2(x)\rangle$, a compact expression for the DeWitt-Schwinger renormalization terms suitable for use in even-dimensional spacetimes is derived. This formula should be useful for calculations of $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ in even dimensions, and the renormalization terms are shown explicitly for four and six dimensions. Furthermore, use of the finite terms of the DeWitt-Schwinger expansion as an approximation to $\langle\phi^2(x)\rangle$ for certain spacetimes is discussed, with application to four and five dimensions.

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I. INTRODUCTION

A. Vacua and particle creation

Hawking radiation shows that a black hole and its horizon are spacetime regions where gravitational effects become important in quantum field theory. The initial approach followed by Hawking [1] was to study quantized ingoing and outgoing field modes in a fixed black hole background. This process yields a flux of particles, produced and emitted out of the surrounding quantum vacuum with a thermal spectrum. In a flat spacetime the vacuum is uniquely defined and relatively easy to find, but in a curved spacetime it is not so simple. There are cases where it may be impossible to define a vacuum state, as in strongly time dependent geometries. On the contrary, there are situations where many vacua exist for a given geometry. The Schwarzschild solution, for example, has three possible vacua: Boulware [2], Unruh [3], and Hartle-Hawking [4], all three of which are interesting and important. The Boulware vacuum describes the vacuum in a region near the surface of a highly compact star, where it has a small negative energy density. Near the horizon of a black hole, however, this energy density blows up, so the Boulware vacuum is inappropriate for a black hole geometry. On the other hand, the Unruh and Hartle-Hawking vacua are both consistent for the Schwarzschild black hole. The Hartle-Hawking vacuum has a small, finite, negative energy at the event horizon, which in turn is responsible for the production of particles and subsequent Hawking radiation at a given temperature. The Unruh vacuum, also having a small, finite, negative energy at the event horizon, is produced by the complete gravitational collapse of an object, and in this sense is more physical. However, the one that usually simplifies the calculations is the Hartle-Hawking vacuum, and, if desired, it is possible to pass from this to the Unruh vacuum via appropriate transformations.

Once a consistent vacuum for the geometry in question, such as a black hole spacetime, has been defined, the associated quantities of interest may be found. For a given field $\phi(x)$ and vacuum $|0\rangle$, where x represents a spacetime point, these quantities may be the vacuum expectation value of the field operator $\phi^2(x)$ i.e. $\langle 0|\phi^2(x)|0\rangle$, or $\langle\phi^2(x)\rangle$ for short, and its associated vacuum expectation value of the stress-energy tensor $T_{\mu\nu}(x)$ i.e. $\langle 0|T_{\mu\nu}(x)|0\rangle$, or $\langle T_{\mu\nu}(x)\rangle$ for short. The quantity $\langle\phi^2(x)\rangle$ is a useful tool in the study of quantum effects in curved spacetimes. When properly renormalized it gives information about vacuum polarization effects and spontaneous symmetry breaking phenomena, although since it is a scalar it does not distinguish between future and past surfaces, such as horizons and infinities. The quantity $\langle T_{\mu\nu}(x)\rangle$ provides information about the energy density and particle production. Moreover, since in general relativity Einstein's equations relate the spacetime curvature to the distribution of matter as encoded in the stress-energy tensor, for quantum fields the expectation value $\langle T_{\mu\nu}(x)\rangle$ is used to determine how the underlying geometry responds to suitable averages of the quantum fields. These back-reaction effects by the quantum fields on the background spacetime are described by the semiclassical Einstein equations $G_{\mu\nu} = 8\pi\langle T_{\mu\nu}(x)\rangle$ (we use $G = 1$, $c = 1$, $\hbar = 1$) (see [5, 6, 7] for reviews and careful explanations). The problem of quantum back reaction is certainly significant in the case of black holes and in other spacetimes with horizons, such as de Sitter spacetime. For instance, in the black hole case it leads to the complete evaporation of the black hole.

B. Renormalizing the vacuum

Since $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ are constructed with product expressions, bilinear in the field operators and evaluated at the same spacetime point, the vacuum expectation value of these quantities diverges. For a theory to have any physical meaning it must give finite results, so some process must be employed to render these quantities finite. Such a process amounts to subtracting off some “unphysical” infinite terms. In flat spacetime, standard normal ordering techniques and other procedures in quantum field theory work well in regularizing and renormalizing the fields, where regularization identifies the infinities, and renormalization eliminates them. In general relativity, however, the energy density itself is a source of curvature. Therefore, when working with a quantum field theory whose energy density is formally divergent we must be very careful about what may be dismissed as unphysical. The standard techniques used in flat spacetimes do not work in curved spacetimes, so one of the great difficulties in understanding quantum processes in a black hole – or any other curved – spacetime is the implementation of consistent regularization and renormalization schemes.

Fortunately, there are several generally accepted consistent, covariant regularization and renormalization schemes for curved spacetimes (see e.g. [5]). Of these, the most widely and consistently used method for $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ calculations is that of isolating the divergent terms of the DeWitt-Schwinger expansion. This covariant geodesic point separation method, developed by Schwinger [8], DeWitt [9, 10], and Christensen [11, 12, 13] (see Barvinsky and Vilkovisky in Ref. [14] for further information), is now usually called the point splitting method. The idea of the point splitting method is that the operator in each product is moved along a geodesic to a nearby spacetime point. The point separated object is expressed in terms of Green's functions $G(x, x')$. In the coincidence limit, where the nearby

spacetime point x' approaches the original point x , there will be terms diverging logarithmically (in even dimensions) and as inverse powers of the point separation. This point splitting method leads naturally to the DeWitt-Schwinger expansion, which gives an approximation for the Green's function $G(x, x')$ when the points x and x' are separated by a small geodesic distance, s , along the shortest geodesic connecting them. The result is actually expanded in powers of the field mass m , with expansion coefficients a_k expressed in terms of geometrical quantities constructed from the Riemann tensor. Other renormalization methods exist, for example, dimensional continuation [15]. Many of these methods have been shown to be equivalent to the DeWitt-Schwinger approach [10].

The divergent terms of the DeWitt-Schwinger expansion are then the renormalizing counter terms to be subtracted from the unrenormalized exact expression for $G(x, x')$, prior to taking the limit $x \rightarrow x'$. Now the expressions for $\langle \phi^2(x) \rangle$ and $\langle T_{\mu\nu}(x) \rangle$, written conveniently in terms of Green's functions, are in fact the properly renormalized expressions $\langle \phi^2(x) \rangle_{\text{ren}}$ and $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$. These renormalized values are those which provide information on spontaneous symmetry breaking and particle production, as well as being essential for calculating the backreaction by quantum fields on the spacetime. This means that semiclassical general relativity has physical meaning when described by the equations $G_{\mu\nu} = 8\pi \langle T_{\mu\nu}(x) \rangle_{\text{ren}}$. With this renormalization process in hand, a complete set of mode functions and their associated creation and annihilation operators must be found by solving the field equations for $\phi(x)$. The vacuum expectation values $\langle \phi^2(x) \rangle_{\text{ren}}$ and $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$ are then put in terms of the field operators. In general the result is a sum over products of mode functions and their derivatives. The sum can, at least in principle, be performed and a finite result is achieved.

C. DeWitt-Schwinger estimates for $\langle \phi^2(x) \rangle$

There is an additional pay-off when using the DeWitt-Schwinger expansion with Christensen's point separation method. Since the expansion is in inverse powers of the field mass m , it is valid for many spacetimes provided m is large enough. In this case the finite terms of the expansion can provide approximations for both $\langle \phi^2(x) \rangle_{\text{ren}}$ and $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$. For instance, given a scalar field $\phi(x)$ the Feynman Green's function corresponds to $\langle \phi^2(x) \rangle$ (see e.g. [5]), so that the finite terms of the expansion directly give a physical $\langle \phi^2(x) \rangle_{\text{ren}}$. The major obstacle in the DeWitt-Schwinger expansion is to compute the coefficients a_k . For a scalar field the first three coefficients, a_0 , a_1 , and a_2 , have been computed by DeWitt [9, 10]; the coefficient a_3 has been computed in the coincidence limit by Gilkey [16]; and, the coefficient a_4 has been computed in the coincidence limit by Avramidi [17], and by Amsterdamski, Berkin, and, O'Connor [18]. Additionally, Barvinsky et. al. [19] have calculated these coefficients using different methods. Thus, Christensen's method plays definitely two roles – it is the basis for point splitting renormalization, and it yields an estimate for the quantities $\langle \phi^2(x) \rangle_{\text{ren}}$ and $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$.

It should be stressed that the DeWitt-Schwinger expansion together with Christensen's point separation method is an approximation that does not hold in all regions of all spacetimes. For example, the results of Kay and Wald [20] show that for a Reissner-Nordström black hole in asymptotically de Sitter spacetime, $\langle \phi^2(x) \rangle$ cannot be regular on both the event and cosmological horizons when these horizons have unequal temperatures.

D. Calculations of $\langle \phi^2(x) \rangle$ and applications

Christensen's work is quite general, and in principle can be applied to any spacetime. For cosmological as well as some black hole applications, see Ref. [5] for works up to around 1980. Many other examples can be given. For massless scalar fields analytical results were reported by several authors. Candelas studied a massless scalar field minimally coupled in the Schwarzschild geometry, where $\langle \phi^2(x) \rangle$ and $\langle T_{\mu\nu}(x) \rangle$ were worked out on the event horizon [21]. Candelas and Howard [22] and Fawcett and Whiting [23] extended the calculation of $\langle \phi^2(x) \rangle$ to the exterior region. Candelas and Jensen [24] calculated $\langle \phi^2(x) \rangle$ in the interior region, and finally Howard and Candelas [25, 26] and Fawcett [27] calculated $\langle T_{\mu\nu}(x) \rangle$ for the whole of Schwarzschild, definitively extending the pioneering work of Candelas [21]. In this context of Schwarzschild black holes it was shown by Hawking [28] and Fawcett and Whiting [23] that the mean square field $\langle \phi^2(x) \rangle$ can give considerable insight into the physical content of the different possible vacua and in the study of theories with spontaneous symmetry breaking. Massless scalar fields in Reissner-Nordström and Kerr-Newman spacetimes were studied by Frolov [29], where $\langle \phi^2(x) \rangle$ was found on the event horizon of a Reissner-Nordström black hole and on the pole of the event horizon of a Kerr-Newman black hole. Since analytical and numerical calculations are difficult, approximation schemes have been devised for calculating $\langle \phi^2(x) \rangle$ and $\langle T_{\mu\nu}(x) \rangle$ for Schwarzschild, Reissner-Nordström, and Kerr-Newman black holes [30, 31, 32, 33, 34]. For massless electromagnetic fields several works have calculated $\langle T_{\mu\nu}(x) \rangle$ for Schwarzschild, Reissner-Nordström, and Kerr-Newman black holes [35, 36, 37, 38]. All these works are for massless fields, where calculations may simplify due to the conformal invariance of the system.

A new approach came with the work of Anderson [39, 40] where the method was applied consistently to massive scalar fields. In Ref. [39] $\langle\phi^2(x)\rangle$ was calculated for a generically coupled massive scalar field in the Schwarzschild geometry, and in [40] a powerful formalism was laid down for finding $\langle\phi^2(x)\rangle$ in a general static spherical geometry, which includes Schwarzschild and Reissner-Nordström solutions. This was possible through the use of the Plana sum formula which converts sums into integrals. The method was extended by Anderson, Hiscock, and Samuel [41, 42] to find $\langle T_{\mu\nu}(x)\rangle_{\text{ren}}$ for a massive scalar field in a general, static, spherical geometry. The approach of Refs. [40, 41, 42] uses a Wentzel-Kramers-Brillouin (WKB) approximation for the mode functions to compute $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ to orders m^{-4} and m^{-2} respectively. It is further found that, when applied to the Reissner-Nordström spacetime, the DeWitt-Schwinger expansion provides values quite close to the numerical results when the field mass $mM \gtrsim 1$, where M is the black hole mass (we put $\hbar = 1$). Anderson's approach has been developed and applied to other cases. Cylindrical black hole spacetimes have been examined by DeBenedictis [43], who worked out $\langle\phi^2(x)\rangle$ for scalar fields, and by Piedra and Oca, who have studied spinor fields [44]. Sushkov [45] studied it for wormholes, Berej and Matyjasek [46] for the spacetime of a nonlinear black hole, Satz, Mazzitelli, and Alvarez [47] for the vacuum outside stars, and Winstanley and Young [48] for lukewarm black holes. Finally, Flachi and Tanaka [49] have used Anderson's method to compute $\langle\phi^2(x)\rangle$ in asymptotically anti-de Sitter black hole geometries. New approaches have been devised by Anderson, Mottola, and Vaulin [50], whereas Popov and Zaslavskii have discussed the WKB approximation in the massless limit [51].

E. Renormalization in d dimensions and this paper

Christensen has remarked [13] that while his methods are valid in arbitrary dimensions, where the procedure is the same as in four dimensions, calculating quantities such as $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ in higher dimensions “would be extremely long and would probably have to be done on a computer,” mainly due to the complexity of the renormalization problem. This comment is still true and the very few works since 1978 that have tried to come to terms with the renormalization techniques in curved d -dimensional spacetimes do prove the difficulty of the extension of the procedure. Nevertheless the interest in these techniques to spacetimes with more than four dimensions has been renewed as the result of progress in areas such as string theory, AdS/CFT (anti-de Sitter/conformal field theory) conjecture, Kaluza-Klein theories, extra large-dimensional scenarios, and the related brane world scenarios.

Earlier, Frolov, Mazzitelli, and Paz [52] studied polarization effects in black hole spacetimes in higher dimensions. In the context of black holes in a braneworld, Casadio [53] discussed back reaction issues. In a very thorough work Decanini and Folacci [54] expressed the DeWitt-Schwinger representation of the Feynman propagator as a Hadamard expansion for even and odd dimensions which clearly exhibit the divergent and the regular parts of the DeWitt-Schwinger representation. In [55, 56] these authors presented the first explicit calculations of the stress-energy tensor in an arbitrary spacetime of d dimensions and provided an expression for $d = 6$ in the large mass limit. Following the ideas developed in Christensen and Fulling [57], Morgan, Thom, Winstanley, and Young [58] have worked out some properties of $\langle T_{\mu\nu}(x)\rangle$ for d -dimensional spherical black holes. Herdeiro, Ribeiro, and Sampaio [59] studied the scalar Casimir effect on a d -dimensional Einstein static universe where renormalization techniques are also used and where, incidentally, the Plana sum formula (also called the Abel-Plana formula) has been applied – in fact the formula was used for the first time in the context of renormalization techniques for the Casimir effect by Mamaev, Mostepanenko, and Starobinsky [60]; see the review [61].

In this paper we use the techniques developed by DeWitt [10], Christensen [11, 12, 13], and Anderson [40, 41, 42] (see also [21, 22]) and apply them to the problem of renormalization of the divergent quantity $\langle\phi^2(x)\rangle$ for a massive scalar field $\phi(x)$ in a d -dimensional static spacetime, carrying out the renormalization by the point splitting technique. The derivations presented by DeWitt [10] and Christensen [11, 12, 13] are quite mathematical in nature, and the end result is not in a form that is amenable for renormalizing $\langle\phi^2(x)\rangle$. The purpose of this paper is to present a compact formula for the renormalization terms that may be applied to $\langle\phi^2(x)\rangle$ calculations, which we achieve for even dimensions. As applications of our results in even dimensions, we single out $d = 4$ and $d = 6$. In $d = 4$ we compare our results with previous results, and surely, it is the most important dimension. We then have chosen $d = 6$ both because it is the simplest case after $d = 4$ and can be consistently realized if one advocates extra large dimension or braneworld scenarios. Odd dimensions may require other methods to find a compact formula for the renormalization terms. In the calculation we also find $\langle\phi^2(x)\rangle_{\text{ren}}$ in first approximation for large enough field masses, in both even and odd dimensions. We give as examples the cases $d = 4$ and $d = 5$. Again, $d = 4$ is singled out because it is the most important dimension and it can be compared immediately with the previous results of other authors, and $d = 5$ is the first odd higher dimension, and could as well be important in scenarios with large extra dimensions. In brief, there are two purposes: one is to kill the divergences in $\langle\phi^2(x)\rangle_{\text{ren}}$; the other is to extract the finite part of $\langle\phi^2(x)\rangle_{\text{ren}}$.

The paper is organized as follows. In Sec. II the calculation of $\langle\phi^2(x)\rangle$ is thoroughly reviewed, including a discussion of the connection between Green's functions and operator theory, and an outline of the standard method for computing

$\langle \phi^2(x) \rangle$ in a static spacetime. This motivates the need for finding a compatible expression for the renormalization terms and shows what form they must take. In Sec. III A the DeWitt-Schwinger expansion for d dimensions is presented, and isolation of the divergent terms is reviewed. In Sec. III B even dimensions are studied. Specifically, in Sec. III B 1, an integral representation for the modified Bessel function $K_\nu(z)$ in the limit of vanishing z is derived for even-dimensional spacetimes. For scalar fields of zero temperature, this integral representation may be used in the expression for the divergent terms. For a scalar field at temperature T , further manipulation is required to make the expression for the divergent terms useful for $\langle \phi^2(x) \rangle$ calculations. The Plana sum formula is employed to convert the integral into a sum plus residual terms, leading to a suitable formula for the renormalization terms in the nonzero temperature case. As an example, the renormalization terms are found for four- and six-dimensional spacetimes in Sec. III B 2. Section IV discusses estimating $\langle \phi^2(x) \rangle$ from the finite terms of the DeWitt-Schwinger expansion, and some concrete examples are given for scalar fields in four- and five-dimensional black hole spacetimes. The results are summarized in Sec. V. In the Appendices A and B we develop some formulas needed in the main part of the work.

II. VACUUM POLARIZATION IN d -DIMENSIONAL STATIC SPACETIMES

A. Green's Function Connection to $\langle \phi^2(x) \rangle$

For a scalar field $\phi(x)$ in a curved spacetime background we start with the action

$$S = \int d^d x \sqrt{|g|} \mathcal{L}, \quad (1)$$

where g is the determinant of the d -dimensional spacetime metric, and \mathcal{L} is the Lagrangian for the scalar field ϕ , given by

$$\mathcal{L} = \frac{1}{2} \{ g^{\mu\nu}(x) \phi(x)_{,\mu} \phi(x)_{,\nu} - [m^2 + \xi R(x)] \phi^2(x) \}. \quad (2)$$

Here m is the mass of the field quanta, and it is assumed there is a coupling between the scalar and gravitational fields of the form $\xi R(x) \phi^2(x)$, where ξ is the coupling constant and $R(x)$ is the Ricci scalar of the background spacetime. Minimal coupling corresponds to $\xi = 0$, while for $\xi = \frac{1}{4} \frac{d-2}{d-1}$ ($\xi = \frac{1}{6}$ in $d = 4$) the field is conformally coupled when $m = 0$, i.e. the action is invariant under conformal transformations of the type $g_{\mu\nu}(x) \rightarrow \bar{g}_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x)$ and $\phi(x) \rightarrow \bar{\phi}(x) = \Omega(x)^{(2-d)/2} \phi(x)$. Varying the action of Eq. (2) in relation to ϕ gives the equation of motion for the field,

$$(\square + m^2 + \xi R(x)) \phi(x) = 0. \quad (3)$$

This is a generalized covariant Klein-Gordon equation, where $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu = |g|^{-1/2} [(|g|)^{1/2} g^{\mu\nu} \phi_{,\nu}]_{,\mu}$ is the Laplace-Beltrami operator in d -dimensional curved spacetime.

Quantization reveals that the field is composed of particles obeying certain commutation relations, and one wants to know how these particles move or propagate in the given curved background spacetime. A propagator is usually defined by modifying the Klein-Gordon equation, Eq. (3), so as to include a source term $J(x)$ such that $(\square + m^2 + \xi R(x)) \phi(x) = J(x)$. This equation may be solved using the standard theory of Green's functions. One of the Green's functions, or propagators, that can be defined is $G_F(x, y)$, which satisfies

$$(\square + m^2 + \xi R(x)) G_F(x, x') = -|g(x)|^{-1/2} \delta^d(x - x'), \quad (4)$$

where $\delta(x)$ is the Dirac delta function. The solution for $\phi(x)$ is then $\phi(x) = \phi_0(x) - \int d^d x' |g(x')|^{1/2} G_F(x, x') J(x')$, where $\phi_0(x)$ is a function that satisfies the Klein-Gordon equation without a source term and $\phi(x)$ corresponds to a quantum field operator acting on some state.

Interestingly, vacuum expectation values of products of field operators can be identified with the Green's function of the wave equation, as we now show. The propagation of a free test particle in a vacuum $|0\rangle$ can be described by the correlation function $G^+(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle \equiv \langle \phi(x) \phi(x') \rangle$, where $\phi(x')$ creates a particle at t' , which in turn is annihilated by $\phi(x)$ at t . This makes sense if $t > t'$. Analogously, the correlation function $G^-(x, x') = \langle \phi(x') \phi(x) \rangle$, describes the propagation of a particle created by $\phi(x)$ at time t , which in turn is annihilated by $\phi(x')$ at t' . This makes sense if $t' > t$. To obtain a correlation function, or propagator, that has physical meaning in relativistic quantum field theory, either $G^+(x, x')$ or $G^-(x, x')$ is used, depending on the sign of the relative time. So to obtain a physically meaningful propagator that combines both $G^+(x, x')$ and $G^-(x, x')$ we can use $\langle T(\phi(x) \phi(x')) \rangle$, where Dyson's time

ordering product operator T is defined as $T(\phi(x)\phi(x')) = \theta(t - t')\phi(x)\phi(x') + \theta(t' - t)\phi(x')\phi(x)$, with $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$. To call $T(\phi(x)\phi(x'))$ the “time ordered product” is apt since the operators occurring under the symbol T are arranged from right to left with increasing times. Such a propagator is called the Feynman propagator, and one can show that this time ordered product of fields is indeed the Feynman Green’s function defined by Eq. (4), i.e.

$$iG_F(x, x') = \langle T(\phi(x)\phi(x')) \rangle. \quad (5)$$

Using the Klein-Gordon equation, Eq. (3), and the properties of the step function $\theta(t' - t)$, one finds

$$(\square_x + m^2 + \xi R(x)) \langle T(\phi(x)\phi(x')) \rangle = -i|g(x)|^{-1/2}\delta^d(x - x'). \quad (6)$$

Care should be taken since the step functions are time-dependent, and instead of zero the result is a distribution $\delta^d(x - x')$ concentrated at equal times. Thus it follows that the vacuum expectation value $\langle T(\phi(x)\phi(x')) \rangle$ is essentially one of the Green’s functions of the covariant generalized Klein-Gordon operator, and we are justified in calling it the Feynman Green’s function $G_F(x, x')$. In other words, the analysis shows that the Feynman propagator is a Green’s function of the Klein-Gordon equation.

Usually in quantum field theory the equation connecting Green’s functions and expectation values, such as Eq. (5), gives a bridge between the theory of propagators, in which scattering amplitudes are written in term of Green’s functions, and the theory of operators, where everything is written in terms of the quantum field $\phi(x)$. One finds the operators and expectation values, thus obtaining the Green’s functions important for interaction theory. We see that in our study, the connection is inverted – we want $\phi^2(x, x')$ at the point x by expressing operator theory in terms of the Green’s function and so we calculate the Green’s function. Thus, Eq. (5) operates as a kind of duality.

Since we are interested in the coincidence limit, Feynman’s Green function is the best to use because it is more physical and also because the boundary conditions allow a Wick rotation of the equation to Euclidean space, where

$$G_F(t, x; t', x') = -iG_E(i\tau, x; i\tau', x'). \quad (7)$$

The Euclidean Green’s function, G_E , now obeys

$$(\square_E - m^2 - \xi R(x)) G_E(x, x') = -|g(x)|^{-1/2}\delta^d(x - x'). \quad (8)$$

where \square_E is now the Laplace-Beltrami operator in d -dimensional curved Euclidean space. There are advantages to working in Euclidean space. For instance, elliptic operators are more easily handled than hyperbolic operators, and after obtaining the Euclidean results one can Wick rotate back to Lorentzian spacetime using Eq. (8) since the boundary conditions for the Feynman propagator are automatically imposed by this procedure.

The Feynman Green’s function, or alternatively the Euclidean Green’s function, is defined in terms of expectation values of products of field operators in the pure vacuum state. This is fine for describing the system at zero temperature. To go further and describe a system at nonzero temperature one has to take into account that the system is no longer in a pure state, it is statistically distributed over all possible states. The full weight of statistical physics must be used, and the Green’s functions are given by the average, suitably weighted, over all pure states of the expectation value of the products of field operators in those pure states (see e.g. Ref. [62]).

B. Calculating the Green’s Function

The standard approach now used for calculating $\langle \phi^2(x) \rangle$ was laid down by Anderson [40], based on earlier works by Candelas and Howard [21, 22]. We start with the Euclidean metric for a static spacetime in d dimensions with line element

$$ds^2 = f(r)d\tau^2 + h(r)dr^2 + r^2d\Omega^2. \quad (9)$$

Here τ is the Euclidean time, $\tau = -it$, r is a kind of radial coordinate, and Ω represents a $(d - 2)$ -dimensional angular space. The only restriction for this method is that the metric must be diagonal. The expectation value $\langle \phi^2(x) \rangle$ is found from the coincidence limit of the Euclidean Green’s function $G_E(x, x) \equiv \lim_{x' \rightarrow x} G_E(x, x')$. For a scalar field in a spacetime given by Eq. (9), $G_E(x, x')$ satisfies [see Eq. (8)]

$$(\square_E - m^2 - \xi R(x)) G_E(x, x') = -\frac{1}{\sqrt{g(x)}}\delta(\tau - \tau')\delta(r - r')\delta(\Omega, \Omega'), \quad (10)$$

where \square_E is the Laplace-Beltrami operator of the Euclidean metric corresponding to Eq. (9). Assuming a separation of variables, the independent homogeneous equations for τ and Ω may be solved. Standard Green's function techniques then tell us that the τ and Ω dependence of $G_E(x, x')$ is equivalent to a representation of the corresponding delta function. We therefore use

$$\delta(\Omega, \Omega') = \sum_{\ell} \sum_{\{\mu_j\}} Y_{\ell, \{\mu_j\}}(\Omega) Y_{\ell, \{\mu_j\}}^*(\Omega'), \quad (11)$$

as the *Ansatz* for the angular dependence of the Euclidean Green's function. The function $Y_{\ell, \{\mu_j\}}$ in Eq. (11) has been generalized to the set of hyperspherical harmonics. In four dimensions these are the usual spherical harmonics such that $\sum_{\ell} \sum_m Y_{\ell, m}(\Omega) Y_{\ell, m}^*(\Omega') = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) P_{\ell}(\Omega \cdot \Omega')$. As the *Ansatz* for the time dependence of the Euclidean Green's function, an integral or a sum representation is used depending on whether the scalar field is at zero or nonzero temperature, respectively. If the scalar field is at zero temperature, then

$$\delta(\tau - \tau') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(\tau - \tau')}. \quad (12)$$

If the scalar field is at nonzero temperature T , then the Green's function is periodic in $\tau - \tau'$ with period T^{-1} [62], and a suitable representation for the delta function is

$$\delta(\tau - \tau') = T \sum_{n=-\infty}^{\infty} \exp[in2\pi T(\tau - \tau')]. \quad (13)$$

Henceforth denote $\varepsilon = \tau - \tau'$. If the scalar field is at zero temperature, then

$$G_E(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega\varepsilon} \sum_{\ell} \sum_{\{\mu_j\}} Y_{\ell, \{\mu_j\}}(\Omega) Y_{\ell, \{\mu_j\}}^*(\Omega') \chi_{\omega\ell}(r, r'). \quad (14)$$

where $\chi_{\omega\ell}(r, r')$ is the last component of the variable separated Green's function – a radial mode function. On the other hand, for a scalar field at some nonzero temperature T

$$G_E(x, x') = \frac{\kappa}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\kappa\varepsilon n} \sum_{\ell} \sum_{\{\mu_j\}} Y_{\ell, \{\mu_j\}}(\Omega) Y_{\ell, \{\mu_j\}}^*(\Omega') \chi_{n\ell}(r, r'), \quad (15)$$

where $\kappa = 2\pi T$. In both cases the radial function obeys a differential equation obtained by putting the above expressions into Eq. (10). Using this expression for the Green's function, and the preceding discussion on the connection between the Green's function and the expectation value $\langle\phi^2(x)\rangle$ allows us to calculate $\langle\phi^2(x)\rangle$ in the Hartle-Hawking vacuum.

There are three difficulties when evaluating these expressions in the coincidence limit. The first is that the equation of motion for the radial function $\chi_{n\ell}$ (equivalently $\chi_{\omega\ell}$) is quite complicated, with exact solutions only available for zero frequency. Asymptotic solutions are obtainable in closed form for massless fields on the horizon of Schwarzschild and Reissner-Nordström black holes [21, 40, 52]. Partially analytical and numerical evaluations of the radial modes occupy the bulk of current research on this topic and will not be discussed here. The other two difficulties are that the sums over both ℓ and n (equivalently the integral over ω) produce divergences. The divergence resulting from the sum over ℓ is actually only an apparent divergence and may be easily remedied. The standard trick is to realize that, given the delta function dependence, we are free to add a term proportional to the delta function. The large ℓ contributions are then eliminated with the help of a WKB approximation. It is the divergence resulting from the sum over n which is more serious, and it is to this matter that we direct our attention.

III. DEWITT-SCHWINGER RENORMALIZATION IN d DIMENSIONS

A. General treatment

To assign a physical meaning to $\langle\phi^2(x)\rangle$, it must be rendered finite via some renormalization process. The divergence resulting from the sum over n in Eqs. (14) and (15) is related to the high frequency behavior of the scalar field. The high frequency modes of the field probe the spacetime geometry in a small neighborhood of an event. Since the metric changes negligibly in this neighborhood an adiabatic, short distance approximation for the propagator should give

the same divergent behavior as Eqs. (14) and (15). Isolating the ultraviolet divergences with such an approximation, these divergent terms can then be subtracted from Eqs. (14) and (15); leaving the renormalized, finite part of the Green's function. The now standard approach is to renormalize the expression for $G_E(x, x')$ via the point splitting method of Christensen applied to the DeWitt-Schwinger expansion of the propagator [8, 10, 12, 13]. In d dimensions, the adiabatic DeWitt-Schwinger expansion of the Euclidean propagator is [13, Eq. 3.10]

$$G_E^{\text{DS}}(x, x') = \frac{\pi \Delta^{1/2}}{(4\pi i)^{d/2}} \sum_{k=0}^{\infty} a_k(x, x') \left(-\frac{\partial}{\partial m^2} \right)^k \left(-\frac{z}{2im^2} \right)^{1-d/2} H_{d/2-1}^{(2)}(z). \quad (16)$$

This equation is slightly different than that found in Ref. [13], where the expression is given for the Feynman Green's function rather than the Euclidean Green's function and uses a different sign convention; the two are related by Eq. (7). Equation (16) introduces several new variables that must be defined. Let $s(x, x')$ be the geodesic distance between x and x' , then define $2\sigma(x, x') = s^2(x, x')$ and $z^2 = -2m^2\sigma(x, x')$. The coefficients $a_k(x, x')$ are generally referred to as DeWitt coefficients. The function $H_\nu^{(2)}(z)$ is a Hankel function of the second kind. Lastly, $\Delta(x, x') = \sqrt{g(x)}D(x, x')\sqrt{g(x')}$ is the Van Vleck-Morette determinant, where $g(x) = \det(g_{\mu\nu}(x))$ and $D(x, x') = \det(-\sigma_{;\mu\nu})$. Expressing $-2m$ and dm^2 in terms of z and dz (for fixed σ), Eq. (16) can be written as

$$G_E^{\text{DS}}(x, x') = \frac{-i\pi\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} a_k(x, x') (-2m^2)^{(d/2-1-k)} z^{-2(d/2-1-k)} \left(\frac{\partial}{z\partial z} \right)^k z^{d/2-1} H_{d/2-1}^{(2)}(z). \quad (17)$$

By the derivative formula for Bessel functions [63],

$$\left(\frac{\partial}{z\partial z} \right)^k z^\mu H_\mu^{(2)}(z) = z^{\mu-k} H_{\mu-k}^{(2)}(z), \quad (18)$$

and defining $\nu = d/2 - 1 - k$, Eq. (17) becomes

$$G_E^{\text{DS}}(x, x') = \frac{-i\pi\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} a_k(x, x') (-2m^2)^\nu z^{-\nu} H_\nu^{(2)}(z). \quad (19)$$

The idea here is that the DeWitt-Schwinger expansion results from a WKB expansion for the Euclidean (or Feynman) propagator for a generic spacetime when the point separation is small. For a particular spacetime this procedure does not give the correct results for the Green's function with finite point separation, but it should reproduce the same divergent terms in the coincidence limit. Therefore, if the divergent terms of the DeWitt-Schwinger expansion can be isolated, then these will be the terms to subtract from $G_E(x, x')$ in order to make it finite as $x \rightarrow x'$.

The Hankel function is related to the usual Bessel functions by $H_\nu^{(2)}(z) = J_\nu - iY_\nu(z)$. Note that $z = i|z|$ is purely imaginary in Euclidean space. For a purely imaginary argument one finds [63]

$$H_\nu^{(2)}(i|z|) = J_\nu(i|z|) - iY_\nu(i|z|) = i^\nu I_\nu(|z|) - i \left[i^{\nu+1} I_\nu(|z|) - \frac{2}{\pi} (-i)^\nu K_\nu(|z|) \right] = 2i^\nu I_\nu(|z|) + \frac{2}{\pi} i(-i)^\nu K_\nu(|z|), \quad (20)$$

leading to

$$G_E^{\text{DS}}(x, x') = \frac{-2i\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} a_k(x, x') (2m^2)^\nu |z|^{-\nu} \left[(-1)^\nu \pi I_\nu(|z|) + iK_\nu(|z|) \right]. \quad (21)$$

Since we are working in Euclideanized space the physical renormalization terms come from the real part of this expression, which will leave only the $K_\nu(|z|)$ terms. The asymptotic behavior of $K_\nu(|z|)$ for small argument, $z \rightarrow 0$, is

$$K_\nu(|z|) \sim \begin{cases} \left(\frac{2}{|z|} \right)^\nu, & \nu > 0, \\ \ln \left(\frac{|z|}{2} \right) + \gamma, & \nu = 0, \\ \left(\frac{2}{|z|} \right)^{-|\nu|}, & \nu < 0, \end{cases} \quad (22)$$

clearly only those terms of the sum for which $\nu \geq 0$ produce divergent terms in the coincidence limit. Let k_d be the largest integer that is less than or equal to $d/2 - 1$; for even dimensions $k_d = (d - 2)/2$ while for odd dimensions

$k_d = (d-3)/2$. It is clear that the divergences arise from those terms for which $k \leq k_d$. Note however, that the terms for $k > d/2 - 1$ are also divergent in the limit of vanishing field mass. For a massless field, where the DeWitt-Schwinger formalism is obviously not well defined, the mass must be replaced in those terms which diverge logarithmically as $m \rightarrow 0$ with a parameter μ which must be fixed by a renormalization condition, or perhaps, as claimed in [10], must ultimately be determined experimentally. Taking the real part of G_E^{DS} , the divergent terms are

$$G_{\text{div}}(x, x') = \frac{2\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} a_k(x, x') (2m^2)^\nu |z|^{-\nu} K_\nu(|z|). \quad (23)$$

The Van Vleck–Morette determinant $\Delta^{1/2}$ and the DeWitt coefficients $a_k(x, x')$ both depend on the point separation z , or equivalently $\sigma(x, x') = \frac{1}{2}\sigma^\rho\sigma_\rho$, where $\sigma_\rho = \sigma_{;\rho}$ [11]. Essentially, σ^ρ is a vector that points from x to x' and has length equal to the distance from x to x' . Consequently, $\sigma^\rho \rightarrow 0$ in the coincidence limit.

Any scalar function may be expanded in a covariant Taylor series of the form [14]

$$f(x') = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \nabla_{\alpha_1} \dots \nabla_{\alpha_k} f(x) \sigma^{\alpha_1} \dots \sigma^{\alpha_k}. \quad (24)$$

Christensen has calculated these expansions for $\Delta^{1/2}$, $a_0(x, x')$, $a_1(x, x')$, and $a_2(x, x')$, which are provided here for completeness [11, 13]. He finds, with $a_0 = 1$,

$$\Delta^{1/2} = 1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta - \frac{1}{24} R_{\alpha\beta;\gamma} \sigma^\alpha \sigma^\beta \sigma^\gamma + \left(\frac{1}{288} R_{\alpha\beta} R_{\gamma\delta} + \frac{1}{360} R_{\alpha}^{\rho}{}^{\tau}{}_{\beta} R_{\rho\gamma\tau\delta} + \frac{1}{80} R_{\alpha\beta;\gamma\delta} \right) \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta + \dots, \quad (25)$$

$$a_1 = \left(\frac{1}{6} - \xi \right) R - \frac{1}{2} \left(\frac{1}{6} - \xi \right) R_{;\alpha} \sigma^\alpha + \left[-\frac{1}{90} R_{\alpha\rho} R_{\beta}^{\rho} + \frac{1}{180} R^{\rho\tau} R_{\rho\alpha\tau\beta} + \frac{1}{180} R_{\rho\tau\kappa\alpha} R^{\rho\tau\kappa}_{\beta} + \frac{1}{120} R_{\alpha\beta;\rho}{}^{\rho} + \left(\frac{1}{40} - \frac{1}{6}\xi \right) R_{;\alpha\beta} \right] \sigma^\alpha \sigma^\beta + \dots, \quad (26)$$

$$a_2 = -\frac{1}{180} R^{\rho\tau} R_{\rho\tau} + \frac{1}{180} R^{\alpha\beta\rho\tau} R_{\alpha\beta\rho\tau} + \frac{1}{6} \left(\frac{1}{5} - \xi \right) R_{;\rho}{}^{\rho} + \frac{1}{2} \left(\frac{1}{6} - \xi \right)^2 R^2 + \dots. \quad (27)$$

For calculations of $\langle \phi^2 \rangle$ up to $d = 7$, only the DeWitt coefficients up to a_2 are required for renormalization. For $d = 8, 9$, a_3 is needed and has been found by Gilkey [16]. For $d = 10, 11$ a_4 is needed and has been calculated in the coincidence limit [17, 18, 19]. For higher dimensional spacetimes, subsequent a_n coefficients must be calculated. Note however that for calculations of $\langle T_{\mu\nu} \rangle$ or other quantities involving derivatives of the field, more of the a_n may be required for a given dimension. Since $\Delta^{1/2}$ and the a_k contain powers of σ^ρ , the entire expression for G_{div} should be expanded in powers of σ^ρ before taking the coincidence limit $x \rightarrow x'$. Let $\Delta^{1/2}$ and a_k be expressed as

$$\Delta^{1/2} = \Delta_0^{1/2} + \Delta_1^{1/2} + \Delta_2^{1/2} + \dots, \quad (28)$$

and

$$a_k = a_k^0 + a_k^1 + a_k^2 + \dots, \quad (29)$$

where the j^{th} numerical index indicates the corresponding term of Eqs. (25)-(27) containing j powers of σ^ρ (note that $\Delta_1^{1/2} = 0$). The bracket notation is the usual notation found in the literature indicating the coincidence limit, e.g. $[a_k] = a_k^0$ is the term containing zero powers of σ^ρ .

These expressions can be put together and expanded to the appropriate order for any dimension. The result, however, would still not be in a form that can be combined with Eqs. (14)-(15). It would therefore be useful to workers in the field to have a compact formula for the renormalization terms as applied to calculations of $\langle \phi^2(x) \rangle$ and $\langle T_{\mu\nu}(x) \rangle$.

The modified Bessel function $K_\nu(|z|)$ behaves differently for even and odd dimensions, so they must be considered separately. In even dimensions ν is an integer while for odd dimensions ν is a half integer. In the small z limit one may verify that $|z|^{-\nu} K_\nu(|z|)$ behaves as

$$|z|^{-\nu} K_\nu(|z|) = \sum_{n=1}^{\nu} \frac{(-1)^{\nu-n} \Gamma(n)}{2^{\nu-2n+1} \Gamma(\nu-n+1) |z|^{2n}} + \frac{(-1)^{\nu}}{2^{\nu+1} \Gamma(\nu+1)} \left(\sum_{n=1}^{\nu} \frac{1}{n} - 2 \left(\ln \frac{|z|}{2} + \gamma \right) \right) + O(|z|) \quad (30)$$

for integer ν , and as

$$|z|^{-\nu} K_\nu(|z|) = \sum_{n=1}^{\nu+\frac{1}{2}} \frac{(-1)^{\nu+n+\frac{1}{2}} 2^{2n-1} \Gamma(n-1/2)}{2^{\nu+1} \Gamma(\nu-n+3/2) |z|^{2n-1}} + \frac{(-1)^{\nu+\frac{1}{2}} \pi}{2^{\nu+1} \Gamma(\nu+1)} + O(|z|) \quad (31)$$

for half-integral ν . Incidentally, from these expansions for $K_\nu(|z|)$ one can begin to see the connection with the Hadamard form of the Green's function [54, 55]. These expressions imply that multiplying $\Delta^{1/2}$, a_k , and $|z|^{-\nu} K_\nu(|z|)$ requires expansions of $\Delta^{1/2}$ and a_k to order 2ν in σ^ρ prior to taking the coincidence limit. Some authors refer to this as the “adiabatic order.” Using the expansions of a_k and $\Delta^{1/2}$, Eqs. (28) and (29), we may collect $a_k \Delta^{1/2}$ in powers of σ^ρ ,

$$a_k \Delta^{1/2} = a_k^0 \Delta_0^{1/2} + (a_k^0 \Delta_1^{1/2} + a_k^1 \Delta_0^{1/2}) + (a_k^0 \Delta_2^{1/2} + a_k^1 \Delta_1^{1/2} + a_k^2 \Delta_0^{1/2}) + \cdots = [a_k] [\Delta^{1/2}] + \sum_{p=1}^{\infty} \sum_{j=0}^p a_k^j \Delta_{p-j}^{1/2}. \quad (32)$$

The summand of Eq. (23) may be expanded explicitly in powers of σ^ρ , giving

$$\begin{aligned} [a_k] [\Delta^{1/2}] (2m^2)^\nu |z|^{-\nu} K_\nu(|z|) &+ (2m^2)^\nu \sum_{n=1}^{\nu} \frac{(-1)^{\nu-n} \Gamma(n)}{2^{\nu-2n+1} \Gamma(\nu-n+1) |z|^{2n}} \sum_{p=1}^{2n} \sum_{j=0}^p a_k^j \Delta_{p-j}^{1/2} \\ &+ (2m^2)^\nu \sum_{n=1}^{\nu} \frac{(-1)^{\nu-n} \Gamma(n)}{2^{\nu-2n+1} \Gamma(\nu-n+1) |z|^{2n}} \sum_{p=2n+1}^{\infty} \sum_{j=0}^p a_k^j \Delta_{p-j}^{1/2} \\ &- \left[\frac{(-1)^\nu}{2^{\nu+1} \Gamma(\nu+1)} \left[2 \left(\ln \frac{|z|}{2} + \gamma \right) - \sum_{n=1}^{\nu} \frac{1}{n} \right] + O(|z|^1) \right] \sum_{p=1}^{\infty} \sum_{j=0}^p a_k^j \Delta_{p-j}^{1/2} \quad (33) \end{aligned}$$

for integral ν (even dimensions), and

$$\begin{aligned} [a_k] [\Delta^{1/2}] (2m^2)^\nu |z|^{-\nu} K_\nu(|z|) &+ (2m^2)^\nu \sum_{n=1}^{\nu+\frac{1}{2}} \frac{(-1)^{\nu+n+\frac{1}{2}} 2^{2n-1} \Gamma(n-1/2)}{2^{\nu+1} \Gamma(\nu-n+3/2) |z|^{2n-1}} \sum_{p=1}^{2n} \sum_{j=0}^p a_k^j \Delta_{p-j}^{1/2} \\ &+ (2m^2)^\nu \sum_{n=1}^{\nu+\frac{1}{2}} \frac{(-1)^{\nu+n+\frac{1}{2}} 2^{2n-1} \Gamma(n-1/2)}{2^{\nu+1} \Gamma(\nu-n+3/2) |z|^{2n-1}} \sum_{p=2n+1}^{\infty} \sum_{j=0}^p a_k^j \Delta_{p-j}^{1/2} + \frac{(-1)^{\nu+\frac{1}{2}} \pi}{2^{\nu+1} \Gamma(\nu+1)} \sum_{p=1}^{\infty} \sum_{j=0}^p a_k^j \Delta_{p-j}^{1/2} \quad (34) \end{aligned}$$

for half-integral ν (odd dimensions). In the second term $a_k \Delta^{1/2}$ has been expanded to order 2ν . Since $a_k^j \Delta_{p-j}^{1/2}$ is proportional to $(\sigma^\alpha)^p$, it is clear that the third and subsequent terms all vanish in the coincidence limit, leaving

$$G_{\text{div}}(x, x') = \frac{2}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} \left[[a_k] (2m^2)^\nu |z|^{-\nu} K_\nu(|z|) + \sum_{n=1}^{\nu} \sum_{p=1}^{2n} \sum_{j=0}^p \frac{2^{2n-1} (-m^2)^{\nu-n} \Gamma(n)}{\Gamma(\nu-n+1)} \frac{a_k^j \Delta_{p-j}^{1/2}}{(\sigma^\rho \sigma_\rho)^n} \right], \quad (35)$$

and

$$G_{\text{div}}(x, x') = \frac{2}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} \left[[a_k] [\Delta^{1/2}] (2m^2)^\nu |z|^{-\nu} K_\nu(|z|) + \sum_{n=1}^{\nu+\frac{1}{2}} \sum_{p=1}^{2n} \sum_{j=0}^p \frac{2^{2n-2} (-m^2)^{\nu+n+\frac{1}{2}} \Gamma(n-\frac{1}{2})}{\Gamma(\nu-n+\frac{3}{2})} \frac{a_k^j \Delta_{p-j}^{1/2}}{(\sigma^\rho \sigma_\rho)^{n-\frac{1}{2}}} \right] \quad (36)$$

for even and odd dimensions, respectively; and where in the second term we have used $|z|^2 = m^2 \sigma^\rho \sigma_\rho$. To reiterate, $k_d = (d-2)/2$ for d even and $k_d = (d-3)/2$ for d odd. While at this stage the even- and odd-dimensional equations appear to have the same form (with the simple replacement $n \rightarrow n - \frac{1}{2}$), it is clear from Eqs. (30)-(31) that the end result is not the same. In particular, as is known, the even-dimensional result contains a logarithmic divergence while the odd dimensional result does not.

The preceding equations are covariant expressions that isolate the divergences in a generic d -dimensional spacetime. To perform any meaningful subtraction of these divergences from the Green's function, these terms must be expressed in a form commensurate with Eqs. (14)-(15). In particular, it would be nice if these terms could be expressed either as an integral over ω or as a sum over n . It will be shown that useful integral and sum representations compatible

ν	Extra Terms
0	0
$\frac{1}{2}$	0
1	$-\frac{c_4}{c_2^2}$
$\frac{3}{2}$	$-\frac{3}{2}\sqrt{\frac{\pi}{2c_2}}\frac{c_4}{\varepsilon c_2^2}$
2	$-\frac{4c_4}{\varepsilon^2 c_2^3} + \frac{m^2 c_4}{c_2^2} + \frac{2}{c_2^2}(3c_4^2 - 2c_2 c_6)$
$\frac{5}{2}$	$-\frac{3}{2}\sqrt{\frac{\pi}{2c_2}}\left[\frac{5c_4}{\varepsilon^3 c_2^3} - \frac{1}{\varepsilon}\left(\frac{m^2 c_4}{c_2^2} + \frac{5}{4c_2^4}(7c_4^2 - 4c_2 c_6)\right)\right]$
3	$-\frac{24c_4}{\varepsilon^4 c_2^5} + \frac{4}{\varepsilon^2 c_2}\left(\frac{m^2 c_4}{c_2^2} + \frac{6}{c_2^3}(2c_4^2 - c_2 c_6)\right) - \frac{m^2}{2}\left(\frac{m^2 c_4}{c_2^2} + \frac{4}{c_2^4}(3c_4^2 - 2c_2 c_6)\right) - \frac{8}{c_2^6}(10c_4^3 - 12c_2 c_4 c_6 + 3c_2^2 c_8)$
$\frac{7}{2}$	$-\frac{3}{2}\sqrt{\frac{\pi}{2c_2}}\left[\frac{35c_4}{\varepsilon^5 c_2^5} - \frac{1}{\varepsilon^3}\left(\frac{5m^2 c_4}{c_2^3} + \frac{35}{4c_2^5}(9c_4^2 - 4c_2 c_6)\right) + \frac{1}{\varepsilon}\left(\frac{m^4 c_4}{2c_2^3} + \frac{5m^2}{4c_2^4}(7c_4^2 - 4c_2 c_6) + \frac{35}{8c_2^6}(33c_4^3 - 36c_2 c_4 c_6 + 8c_2^2 c_8)\right)\right]$
4	$-\frac{192c_4}{\varepsilon^6 c_2^7} + \frac{24}{\varepsilon^4 c_2^3}\left(\frac{m^2 c_4}{c_2^2} + \frac{4}{c_2^3}(5c_4^2 - 2c_2 c_6)\right) - \frac{2}{\varepsilon^2 c_2}\left[m^2\left(\frac{m^2 c_4^2}{c_2^2} + \frac{12}{c_2^4}(2c_4^2 - c_2 c_6)\right) + \frac{96}{c_2^6}(5c_4^3 - 5c_2 c_4 c_6 + c_2^2 c_8)\right]$ $+ \frac{m^2}{3}\left[\frac{m^2}{2}\left(\frac{m^2 c_4}{c_2^2} + \frac{6}{c_2^3}(3c_4^2 - 2c_2 c_6)\right) + \frac{24}{c_2^6}(10c_4^3 - 12c_2 c_4 c_6 + 3c_2^2 c_8)\right] + \frac{48}{c_2^8}(35c_4^4 - 60c_2 c_4^2 c_6 + 20c_2^2 c_4 c_8 + 10c_2^2 c_6^2 - 4c_2^3 c_{10})$

TABLE I: This table shows the extra terms generated by the modified Bessel function $K_\nu(|z|)$ when one makes the replacement $|z|^2 \rightarrow 2m^2 \sum_{n=1}^{\infty} c_{2n} \varepsilon^{2n}$. Integer values of ν are applicable to even-dimensional spacetimes, whereas half-integer values of ν are applicable to odd-dimensional spacetimes.

with Eqs. (14)-(15) can be found for even-dimensional spacetimes. At this time, however, a correspondingly suitable expression for use with odd-dimensional spacetimes remains elusive. Consequently, in what follows we primarily address renormalization with respect to even-dimensional spacetimes.

Unfortunately it does not seem to be possible to obtain a simple, compact, general expression as an integral over ω or sum over n . The first problem is that, while the second term of Eqs. (35)-(36) may simply be finite, as is the case for four dimensions, this is not generally true for higher dimensions, as will be shown explicitly for the six dimensional case below. These additional divergent terms may be addressed by Howard's method [25], described in Appendix A and used below.

As for the first term, we may proceed a little further but must use some care. Recall that the physical parameter approaching zero is $\varepsilon = \tau - \tau'$, then z must be expanded in powers of ε with the end result that $z^2 = -2m^2 \sum_{n=1}^{\infty} c_{2n} \varepsilon^{2n}$ for some r -dependent coefficients c_{2n} . The c_{2n} are combinations of the metric functions f and h , and their derivatives. Expanding z^{-2n} in powers of ε one gets a series of terms proportional to $\varepsilon^{-n}, \varepsilon^{-n+1}, \dots, \varepsilon^{-1}$ plus a constant term. This means that

$$(2m^2)^\nu |z|^{-\nu} K_\nu(|z|) = \frac{(2m^2)^\nu}{(m\varepsilon\sqrt{f})^\nu} K_\nu(m\varepsilon\sqrt{f}) + \text{Extra Terms.} \quad (37)$$

The extra terms, which will be denoted E_ν , must be determined for each ν , and so far a compact expression giving the extra terms for a given ν is unavailable, but may possibly be found from a lengthy exercise in combinatorial gymnastics. The extra terms for the first few integral and half-integral ν are presented in Tables I and expressed in terms of the coefficients c_{2n} .

In practice the extra terms are straightforward to calculate using a computer algebra system. One simply takes the difference of $(2m^2)^\nu |z|^{-\nu} K_\nu(|z|)$, with $|z|^2 \rightarrow 2m^2 \sum_{n=1}^{\infty} c_{2n} \varepsilon^{2n}$ and expanded around $\varepsilon = 0$, and $(2m^2)^\nu |z|^{-\nu} K_\nu(|z|)$ with the replacement $|z|^2 \rightarrow 2m^2 c_2 \varepsilon^2$. One important point that should be noted is that finding the coefficient c_{2n} in the expansion of $|z|$ requires one to first calculate σ^μ to order ε^{2n-1} . In Table II the extra terms are presented explicitly in terms of the metric functions f and h for the first few integral and half-integral values of ν . In four dimensions $k_d = 1$, in which case ν ranges from 0 to 1 and these extra terms contribute no new divergences. In six dimensions $k_d = 2$, ν ranges from 0 to 2, so one extra divergent term arises when $\nu = 2$. In eight dimensions the extra divergences come from both $\nu = 2$ and $\nu = 3$. Obviously a similar situation occurs for odd dimensions.

All these divergent terms lurking about within Eqs. (35)-(36) must now be expressed as an integral over ω or a sum over n , commensurate with Eqs. (14)-(15). To this end we use an integral representation of the modified Bessel function, some identities proved by Howard [25], and the Plana sum formula [40, 64, 65, 66],

$$\int_j^\infty f(n)dn = \sum_{n=j}^\infty f(n) - \frac{1}{2}f(j) - i \int_0^\infty \frac{dt}{e^{2\pi t} - 1} [f(j+it) - f(j-it)], \quad (38)$$

to convert between integrals and sums.

ν	Extra Terms (E_ν)
0	0
$\frac{1}{2}$	0
1	$\frac{f'^2}{24f^2h}$
$\frac{3}{2}$	$\frac{\sqrt{\pi}f'^2}{16\varepsilon f^{5/2}h}$
2	$\frac{f'^2}{f^2h} \left[\frac{m^2}{3} \left(\frac{1}{m^2\varepsilon^2f} - \frac{1}{8} \right) + \frac{1}{60h} \left(\frac{23f'^2}{24f^2} + \frac{f'h'}{2fh} - \frac{f''}{f} \right) \right]$
$\frac{5}{2}$	$\frac{\sqrt{\pi}f'^2}{8f^{5/2}h} \left(\frac{5}{\varepsilon^3f} - \frac{1}{\varepsilon} \left(\frac{m^2}{2} - \frac{f'}{4fh} \left(\frac{17f'}{16f} + \frac{h'}{2h} - \frac{f''}{f'} \right) \right) \right)$
3	$\frac{f'^2}{f^2h} \left[m^4 \left(\frac{4}{m^4\varepsilon^4f^2} - \frac{1}{3m^2\varepsilon^2f} + \frac{1}{48} \right) + \frac{m^2}{5} \frac{f'}{fh} \left(\frac{1}{m^2\varepsilon^2f} \left(\frac{7f'}{6f} + \frac{h'}{2h} - \frac{f''}{f'} \right) - \frac{1}{12} \left(\frac{23f'}{24f} + \frac{h'}{2h} - \frac{f''}{f'} \right) \right) \right. \right.$ $\left. + \frac{f'}{f^2h^2} \left(-\frac{f'}{12} \left(\frac{23f'^2}{360f^2} + \frac{f'h'}{5fh} + \frac{h'^2}{16h^2} \right) + \frac{f''}{6} \left(\frac{f'}{5f} + \frac{h'}{8h} - \frac{f''}{8f'} \right) \right) \right]$

TABLE II: This table shows the extra terms generated by the modified Bessel function $K_\nu(|z|)$ for the specific metric of Eq. (9).

B. DeWitt-Schwinger Renormalization in Even dimensions

1. Renormalization formulas at zero and nonzero temperatures

For ν an integer it is shown in Appendix B that an integral representation of $K_\nu(z)$ for small z is

$$K_\nu(z) = \frac{(-1)^\nu \sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2} \right)^\nu \int_0^\infty dt \cos(zt) (t^2 + 1)^{\nu-1/2}. \quad (39)$$

For $T = 0$ one has to connect Eq. (39) with the $e^{i\omega\varepsilon}$ dependence of Eq. (14). Consider the change of variables $t = \omega/\sqrt{m^2f}$, and $z = m\varepsilon\sqrt{f}$; then

$$\frac{(2m^2)^\nu}{(m\varepsilon\sqrt{f})^\nu} K_\nu(m\varepsilon\sqrt{f}) = \frac{\sqrt{\pi}}{(-f)^\nu \Gamma(\nu + \frac{1}{2})} \int_0^\infty \cos(\varepsilon\omega) (\omega^2 + m^2f)^{\nu-1/2} d\omega. \quad (40)$$

This result generalizes the integral representation found by Anderson [40, Eq. (3.4a) and (3.4b)].

For nonzero temperature T , one has to connect Eq. (39) with the $e^{i\kappa\varepsilon n}$ dependence of Eq. (15). We instead make the change of variables $t = x\kappa/\sqrt{m^2f}$ to first obtain

$$\frac{(2m^2)^\nu}{(m\varepsilon\sqrt{f})^\nu} K_\nu(m\varepsilon\sqrt{f}) = \frac{\kappa\sqrt{\pi}}{(-f)^\nu \Gamma(\nu + \frac{1}{2})} \int_0^\infty \cos(\kappa\varepsilon x) (\kappa^2 x^2 + m^2f)^{\nu-1/2} dx. \quad (41)$$

The Plana sum formula, Eq. (38), enables the integral in this equation to be converted into a sum plus some residues and is valid if the function f satisfies three conditions: (i) $f(\tau + it)$ is holomorphic for $\tau \geq j$ for any t , (ii) $\lim_{t \rightarrow \infty} |f(\tau + it)| e^{-2\pi|t|} = 0$ uniformly for every $\tau \geq j$, and (iii) $\lim_{\tau \rightarrow \infty} \int_{-\infty}^\infty dt |f(\tau + it)| e^{-2\pi|t|} = 0$. A naive application of the Plana sum formula would be to use $j = 0$, corresponding to the lower limit of integration in Eq. (41). However, for $j = 0$ the integrand of Eq. (41) is not holomorphic at $\tau = 0$. Consequently, one must break up the integral into two parts

$$\int_0^\infty dx \cos(\kappa\varepsilon x) (\kappa^2 x^2 + m^2f)^{\nu-1/2} = \int_0^1 dx \cos(\kappa\varepsilon x) (\kappa^2 x^2 + m^2f)^{\nu-1/2} + \int_1^\infty dx \cos(\kappa\varepsilon x) (\kappa^2 x^2 + m^2f)^{\nu-1/2}. \quad (42)$$

For the first integral $\cos(\kappa\varepsilon x) \approx 1$ in the coincidence limit and the solution may be expressed as a hypergeometric function depending on ν [67, Eq. (2.271)],

$$\int_0^1 dx \cos(\kappa\varepsilon x) (\kappa^2 x^2 + m^2f)^{\nu-1/2} = (m^2f)^{\nu-1/2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2} - \nu, \frac{3}{2}, -\frac{\kappa^2}{m^2f} \right). \quad (43)$$

In general, this hypergeometric function is equivalent to a polynomial in half integer powers of $(\kappa^2 + m^2f)$ plus a logarithmic term. Applying the Plana sum formula to the second integral gives

$$\begin{aligned} \int_1^\infty \cos(\kappa\varepsilon x) (\kappa^2 x^2 + m^2f)^{\nu-1/2} dx &= \sum_{n=1}^\infty \cos(\kappa\varepsilon n) (\kappa^2 n^2 + m^2f)^{\nu-\frac{1}{2}} - \frac{1}{2} (\kappa^2 + m^2f)^{\nu-\frac{1}{2}} \\ &\quad - i \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \left\{ [(1+it)^2 \kappa^2 + m^2f]^{\nu-1/2} - [(1-it)^2 \kappa^2 + m^2f]^{\nu-1/2} \right\}. \end{aligned} \quad (44)$$

Putting this together, we have

$$\begin{aligned} & \frac{(2m^2)^\nu}{(m\varepsilon\sqrt{f})^\nu} K_\nu(m\varepsilon\sqrt{f}) \\ &= \frac{\kappa\sqrt{\pi}}{(-f)^\nu\Gamma(\nu+\frac{1}{2})} \left\{ \sum_{n=1}^{\infty} \cos(\kappa\varepsilon n) (\kappa^2 n^2 + m^2 f)^{\nu-\frac{1}{2}} - \frac{1}{2}(\kappa^2 + m^2 f)^{\nu-\frac{1}{2}} + (m^2 f)^{\nu-\frac{1}{2}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \nu, \frac{3}{2}, -\frac{\kappa^2}{m^2 f}\right) \right. \\ & \quad \left. - i \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \left\{ [(1+it)^2 \kappa^2 + m^2 f]^{\nu-1/2} - [(1-it)^2 \kappa^2 + m^2 f]^{\nu-1/2} \right\} \right\}. \quad (45) \end{aligned}$$

This result generalizes the sum representation found by Anderson [40, Eq. (3.7a) and (3.7b)]. Finally, the renormalization terms for the d -dimensional spacetime of Eq. (9) are

$$\begin{aligned} G_{\text{div}}(x, x') &= \frac{2}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} \left[\frac{[a_k] \sqrt{\pi}}{(-f)^\nu \Gamma(\nu + \frac{1}{2})} \int_0^\infty \cos(\omega\varepsilon) (\omega^2 + m^2 f)^{\nu-1/2} d\omega \right. \\ & \quad \left. + [a_k] E_\nu + \sum_{n=1}^\nu \sum_{p=1}^{2n} \sum_{j=0}^p \frac{2^{2n-1} (-m^2)^{\nu-n} \Gamma(n)}{\Gamma(\nu-n+1)} \frac{a_k^j \Delta_{p-j}^{1/2}}{(\sigma^\rho \sigma_\rho)^n} \right] \quad (46) \end{aligned}$$

for the case of a scalar field at zero temperature $T = 0$, and

$$\begin{aligned} G_{\text{div}}(x, x') &= \frac{2}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} \left\{ \frac{[a_k] \kappa \sqrt{\pi}}{(-f)^\nu \Gamma(\nu + \frac{1}{2})} \left[\sum_{n=1}^{\infty} \cos(\kappa\varepsilon n) (\kappa^2 n^2 + m^2 f)^{\nu-\frac{1}{2}} - \frac{1}{2}(\kappa^2 + m^2 f)^{\nu-\frac{1}{2}} \right. \right. \\ & \quad \left. \left. - i \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \left\{ [(1+it)^2 \kappa^2 + m^2 f]^{\nu-1/2} - [(1-it)^2 \kappa^2 + m^2 f]^{\nu-1/2} \right\} \right] \right. \\ & \quad \left. + (m^2 f)^{\nu-\frac{1}{2}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \nu, \frac{3}{2}, -\frac{\kappa^2}{m^2 f}\right) \right\} + [a_k] E_\nu + \sum_{n=1}^\nu \sum_{p=1}^{2n} \sum_{j=0}^p \frac{2^{2n-1} (-m^2)^{\nu-n} \Gamma(n)}{\Gamma(\nu-n+1)} \frac{a_k^j \Delta_{p-j}^{1/2}}{(\sigma^\rho \sigma_\rho)^n} \quad (47) \end{aligned}$$

for a scalar field at nonzero temperature $T > 0$.

2. Examples: $d = 4$ and $d = 6$

The formulas given above provide simple expressions to calculate the renormalization terms for the generic even-dimensional spacetime of Eq. (9). Below we mention the case $d = 4$ and study more carefully the case $d = 6$. For any d -dimensional spacetime with line element given by Eq. (9), we generalize σ^μ to [11, 42]

$$\sigma^\tau = -\varepsilon + \frac{\varepsilon^3}{24} \frac{(f')^2}{fh} + \frac{\varepsilon^5}{120} \left(\frac{f'^4}{8f^2 h^2} + \frac{3}{16} \frac{(f')^3 h'}{fh^3} - \frac{3}{8} \frac{f'^2 f''}{fh^2} \right) + O(\varepsilon^7) \quad (48a)$$

$$\sigma^r = \frac{\varepsilon^2 f'}{4h} - \frac{\varepsilon^4}{24} \left(-\frac{f'^2 h'}{8h^3} + \frac{f' f''}{4h^2} \right) + O(\varepsilon^6) \quad (48b)$$

$$\sigma^{\theta_i} = 0 \quad i = 1 \dots d-2. \quad (48c)$$

In applying Eqs. (46) and (47), we use the values for E_ν as listed in Table II.

$d = 4$:

It is straightforward to show that in four dimensions Eqs. (46) and (47) are identical to those obtained by Anderson [40, 42] and used by several subsequent authors. After letting $\varepsilon \rightarrow 0$ we find, for $T = 0$,

$$G_{\text{div}}(x, x') = -\frac{1}{4\pi^2} \int_0^\infty d\omega \left[\frac{1}{f} (\omega^2 + m^2 f)^{1/2} + \frac{1}{2} \left(\xi - \frac{1}{6} \right) R (\omega^2 + m^2 f)^{-1/2} \right] - \frac{f'}{192fh} \left(\frac{4}{r} + \frac{2f''}{f'} - \frac{2f'}{f} - \frac{h'}{h} \right). \quad (49)$$

One may verify that expanding Eq. (47) correctly reproduces the results obtained by Anderson for $T > 0$.

$d = 6$:

Another, less trivial, example can be given for a scalar field in six dimensions. We consider three classes of spacetime: spherical, flat, and hyperbolic, corresponding to $\mathfrak{K} = 1, 0$, or -1 , respectively. Consider first the last terms of Eqs. (46) and (47). Using the values of E_ν given in Table II and calculating the last sums, we find

$$\sum_{k=0}^{k_d} \left[[a_k] E_\nu + \sum_{n=1}^{\nu} \sum_{p=1}^{2n} \sum_{j=0}^p \frac{2^{2n-1} (-m^2)^{\nu-n} \Gamma(n)}{\Gamma(\nu-n+1)} \frac{a_k^j \Delta_{p-j}^{1/2}}{(\sigma^\rho \sigma_\rho)^n} \right] = -\frac{1}{\varepsilon^2} \frac{f'}{6f^2 h} \left(\frac{8}{r} + \frac{2f''}{f'} - \frac{3f'}{f} - \frac{h'}{h} \right) + C_6^{\mathfrak{K}} \quad (50)$$

where

$$\begin{aligned} C_6^{\mathfrak{K}} = m^2 \frac{f'}{24fh} & \left(\frac{8}{r} + \frac{2f''}{f'} - \frac{2f'}{f} - \frac{h'}{h} \right) + \frac{f'}{fh^2} \left\{ \frac{2}{r^3} \left[\left(\frac{1}{5} - \xi \right) - \mathfrak{K}h \left(\frac{1}{6} - \xi \right) \right] \right. \\ & + \frac{1}{r^2} \left[\mathfrak{K}h \left(\frac{1}{6} - \xi \right) \left(\frac{f'}{f} + \frac{h'}{2h} - \frac{f''}{f'} \right) + \frac{f'}{12f} - \frac{5h'}{2h} \left(\frac{4}{25} - \xi \right) + \frac{f''}{f'} \left(\frac{1}{10} - \xi \right) \right] \\ & + \frac{1}{r} \left[-\frac{5f'^2}{6f^2} \left(\frac{9}{25} - \xi \right) - \frac{f'h'}{3fh} \left(\frac{3}{5} - \xi \right) + \frac{5h'^2}{6h^2} \left(\frac{3}{50} - \xi \right) + \frac{f''}{f'} \left(\frac{f'}{f} \left(\frac{73}{180} - \xi \right) + \frac{h'}{3h} \left(\frac{1}{5} + \xi \right) \right) \right. \\ & - \frac{h''}{3h} \left(\frac{1}{20} - \xi \right) - \frac{f'''}{15f'} \left. + \frac{f'}{f} \left(\frac{f'^2}{8f^2} \left(\frac{27}{40} - \xi \right) - \frac{7f'h'}{48fh} \left(\frac{127}{210} - \xi \right) + \frac{5h'^2}{48h^2} \left(\frac{63}{100} - \xi \right) \right] \right. \\ & - \frac{f''}{f} \left(\frac{7f'}{24f} \left(\frac{127}{210} - \xi \right) + \frac{5h'}{24h} \left(\frac{61}{100} - \xi \right) - \frac{f''}{12f'} \left(\frac{11}{20} - \xi \right) \right) - \frac{f'}{12f} \left(\frac{13}{20} - \xi \right) \left(\frac{h''}{2h} - \frac{f'''}{f'} \right) \\ & \left. + \frac{7h'^3}{240h^3} - \frac{19f''h'^2}{480f'h^2} - \frac{13h'h''}{480h^2} + \frac{f''h''}{60f'h} + \frac{h'f'''}{40f'h} + \frac{h'''}{240h} - \frac{f^{(4)}}{120f'} \right\}. \quad (51) \end{aligned}$$

Using Eq. (A5) for the integral representation of ε^{-2} , the $T = 0$ divergent terms in the limit $\varepsilon \rightarrow 0$ are

$$G_{\text{div}}(x, x') = \frac{2}{(4\pi)^3} \left\{ \int_0^\infty d\omega \left[\frac{4}{3f^2} (\omega^2 + m^2 f)^{3/2} + \frac{2}{f} \left(\xi - \frac{1}{6} \right) R(\omega^2 + m^2 f)^{1/2} + [a_2] (\omega^2 + m^2 f)^{-1/2} \right. \right. \\ \left. \left. + \frac{f'\omega}{6f^2 h} \left(\frac{8}{r} + \frac{2f''}{f'} - \frac{3f'}{f} - \frac{h'}{h} \right) \right] + C_6^{\mathfrak{K}} \right\} \quad (52)$$

where, for $d = 6$,

$$\begin{aligned} [a_2] = \frac{1}{240h^2} & \left\{ \frac{16}{r^4} (\mathfrak{K}h - 1) [(-41 + 420\xi - 1080\xi^2) + \mathfrak{K}h (29 - 360\xi + 1080\xi^2)] \right. \\ & + \frac{16}{r^3} \left(\frac{f'}{f} - \frac{h'}{h} \right) [(29 - 290\xi + 720\xi^2) - 5kh (5 - 54\xi + 144\xi^2)] \\ & + \frac{4}{r^2} \left[10\mathfrak{K}(1 - 6\xi)^2 \frac{f'h}{f} \left(\frac{f'}{f} + \frac{h'}{h} - 2\frac{f''}{f'} \right) + (23 - 140\xi + 120\xi^2) \frac{f'^2}{f^2} - 4(4 - 85\xi + 330\xi^2) \frac{f'}{f} \frac{h'}{h} \right. \\ & + 5(-15 + 56\xi + 96\xi^2) \frac{h'^2}{h^2} + 4(1 - 40\xi + 180\xi^2) \frac{f''}{f} - 40(-1 + 5\xi) \frac{h''}{h} \left. \right] \\ & + \frac{4}{r} \left[\frac{f'}{f} \left(-(23 - 140\xi + 120\xi^2) \frac{f'^2}{f^2} + (-33 + 140\xi + 120\xi^2) \frac{h'^2}{h^2} \right) \right. \\ & + 2\frac{f''}{f} \left((21 - 130\xi + 120\xi^2) \frac{f'}{f} - (-13 + 40\xi + 120\xi^2) \frac{h'}{h} \right) \\ & - 4(1 - 5\xi) \left(\frac{h'}{h} \left(6\frac{f'^2}{f^2} - 14\frac{h'^2}{h^2} \right) - \frac{h''}{h} \left(4\frac{f'}{f} - 13\frac{h'}{h} \right) + 4\frac{f^{(3)}}{f} - 2\frac{h^{(3)}}{h} \right) \left. \right] \\ & + \frac{f'^2}{f^2} \left((21 - 110\xi + 30\xi^2) \frac{f'^2}{f^2} - 2(-13 + 70\xi - 30\xi^2) \frac{f'}{f} \frac{h'}{h} + 5(5 - 26\xi + 6\xi^2) \frac{h'^2}{h^2} \right) \\ & - 2\frac{f'f''}{f^2} \left(2(13 - 70\xi + 30\xi^2) \frac{f'}{f} + 5(5 - 27\xi + 12\xi^2) \frac{h'}{h} - 10(1 - 6\xi + 6\xi^2) \frac{f''}{f'} \right) \\ & \left. + 2(1 - 5\xi) \left(14\frac{f'}{f} \frac{h'^3}{h^3} - 19\frac{h'^2}{h^2} \frac{f''}{f} - \frac{f'}{f} \frac{h''}{h} \left(5\frac{f'}{f} + 13\frac{h'}{h} \right) + 8\frac{f''}{f} \frac{h''}{h} + 2\frac{f^{(3)}}{f} \left(5\frac{f'}{f} + 6\frac{h'}{h} \right) + 2\frac{f'}{f} \frac{h^{(3)}}{h} - 4\frac{f^{(4)}}{f} \right) \right\}. \quad (53) \end{aligned}$$

On the other hand, for nonzero temperature $T > 0$, Eq. (A1) may be employed to find a sum representation of ε^{-2} and one may show that

$$\begin{aligned}
G_{\text{div}}(x, x') = & \frac{2}{(4\pi)^3} \left\{ \kappa \sum_{n=1}^{\infty} \left[\frac{4}{3f^2} (\kappa^2 n^2 + m^2 f)^{3/2} + \frac{2}{f} (\xi - \frac{1}{6}) R (\kappa^2 n^2 + m^2 f)^{1/2} + [a_2] (\kappa^2 n^2 + m^2 f)^{-1/2} \right. \right. \\
& + \frac{n\kappa f'}{6f^2 h} \left(\frac{8}{r} + \frac{2f''}{f'} - \frac{3f'}{f} - \frac{h'}{h} \right) \Big] + \ln \left(\frac{\kappa + \sqrt{\kappa^2 + m^2 f}}{\sqrt{m^2 f}} \right) \left[\frac{1}{2} m^4 + (\xi - \frac{1}{6}) m^2 R + [a_2] \right] \\
& - \frac{\kappa}{6f^2} (2\kappa^2 - m^2 f) (\kappa^2 + m^2 f)^{1/2} - \frac{\kappa}{2} [a_2] (\kappa^2 + m^2 f)^{-1/2} + \frac{\kappa^2}{12} + C_6^{\mathfrak{R}} \\
& - \frac{4i\kappa}{3f^2} \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} \left\{ [(1+it)^2 \kappa^2 + m^2 f]^{3/2} - [(1-it)^2 \kappa^2 + m^2 f]^{3/2} \right\} \\
& - \frac{2i\kappa}{f} (\xi - \frac{1}{6}) R \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} \left\{ [(1+it)^2 \kappa^2 + m^2 f]^{1/2} - [(1-it)^2 \kappa^2 + m^2 f]^{1/2} \right\} \\
& \left. - i\kappa [a_2] \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} \left\{ [(1+it)^2 \kappa^2 + m^2 f]^{-1/2} - [(1-it)^2 \kappa^2 + m^2 f]^{-1/2} \right\} \right\}. \quad (54)
\end{aligned}$$

These results are quite general and unwieldy, but expressions such as Eq. (53), for $[a_2]$, may become quite simple for particular situations. For a minimally coupled scalar field in a six-dimensional, asymptotically anti-de Sitter Reissner-Nordström black hole spacetime with

$$f = \mathfrak{R} - \frac{\Lambda}{3} r^2 - \frac{M}{r^3} + \frac{Q^2}{r^6}, \quad (55)$$

$[a_2]$ reduces to

$$[a_2] = \frac{4(14Q^2 - 5Mr^3)^2}{75r^{16}}; \quad (56)$$

a remarkably simple result (note that in six dimensions a minimally coupled field has $\xi = 1/5$).

Put in perspective, we have presented a formula for the renormalization terms of $\langle \phi^2(x) \rangle$, which in the particular important case of a minimally coupled scalar field in a $(d=6)$ -dimensional asymptotically anti-de Sitter Reissner-Nordström black hole spacetime, yields an extremely simple compact formula. The dimension $d=6$ is of importance because it is the simplest even higher dimension that can be made compatible with the extra large dimension or brane world scenarios.

IV. ESTIMATE OF $\langle \phi^2(x) \rangle$ FOR MASSIVE FIELDS

Using the DeWitt-Schwinger expansion we have isolated the divergent terms in the coincidence limit. These were precisely the terms of the expansion up to $k = k_d$. The DeWitt-Schwinger expansion does not give the correct results for $\langle \phi^2(x) \rangle$, even after removing the divergences, because the expansion depends only on the local structure of the spacetime whereas the true field modes also depend in part on the global structure of the spacetime, for example, the effective potential around a black hole. However, when the field is massive enough, the higher order terms in the expansion may be used as an approximation to the renormalized value of $\langle \phi^2(x) \rangle$ such that

$$\langle \phi^2(x) \rangle \approx \lim_{x' \rightarrow x} G_{\text{ren}}(x, x') = \lim_{z \rightarrow 0} \frac{2}{(4\pi)^{d/2}} \sum_{k > k_d} \Delta^{1/2} a_k (2m^2)^{\nu} |z|^{-\nu} K_{\nu}(|z|). \quad (57)$$

For $k > k_d$, $\nu < 0$, in which case $|z|^{-\nu} K_{\nu}(|z|) = 2^{-(\nu+1)} \Gamma(-\nu) + O(z^1)$ for small z in both even and odd dimensions. It follows that

$$\langle \phi^2(x) \rangle \approx \frac{1}{(4\pi)^{d/2}} \sum_{k > k_d} [a_k] m^{2\nu} \Gamma(-\nu) = \begin{cases} (2^d \pi^{d/2} m^2)^{-1} [a_{\frac{d}{2}}] + \dots & d \text{ even}, \\ (2^d \pi^{(d-1)/2} m)^{-1} [a_{\frac{d-1}{2}}] + \dots & d \text{ odd}, \end{cases} \quad (58)$$

in the coincidence limit.

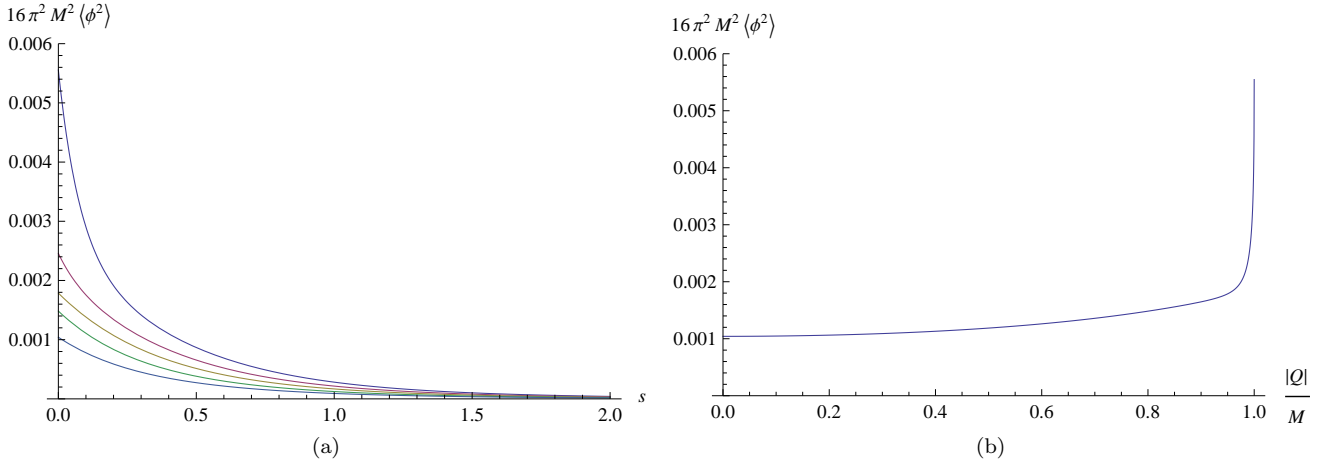


FIG. 1: Plot of $\langle\phi^2(s)\rangle$ in a four-dimensional Reissner-Nordström black hole spacetime with $mM = 2$. (a) Near the horizon. From top to bottom the curves correspond to the cases $|Q|/M = 1.0, 0.99, 0.95, 0.8, 0.0$. The radial coordinate $s = r/M - 1 - \sqrt{1 - (Q/M)^2}$ is zero at the horizon. (b) On the horizon as a function of $|Q|/M$.

$d = 4$:

For $d = 4$ one finds $\langle\phi^2(x)\rangle_{d=4} \approx (4\pi m)^{-2}[a_2]$. This agrees exactly with the result reported by Anderson [40, Eq. (4.1)], but note that Anderson's equation contains a typographical misprint in the first term, which should be $\frac{1}{6}(\xi - \frac{1}{5})R_{;\rho}{}^{;\rho}$. This misprint has no consequence since $R = 0$ in the spacetime considered by Anderson, but would be important elsewhere. Calculating the coefficient $[a_2]$ for a Reissner-Nordström black hole spacetime, for which

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (59)$$

gives

$$[a_2] = \frac{1}{45r^8}(13Q^4 - 24MQ^2r + 12M^2r^2), \quad (60)$$

leading to the near horizon behavior of $\langle\phi^2(x)\rangle_{d=4}$ plotted in Fig. 1(a). It may be seen that this correctly reproduces Fig. 3 of Ref. [40]. Figure 1(b) shows the behavior of $\langle\phi^2(x)\rangle_{d=4}$ on the horizon as a function of the charge-to-mass ratio, where it can be seen that the value of $\langle\phi^2(x)\rangle_{d=4}$ increases rapidly to a finite value as the black hole approaches extremality.

In Refs. [40, 42] it is emphasized that the finite terms of the DeWitt-Schwinger expansion give a good estimate for $\langle\phi^2(x)\rangle_{\text{ren}}$ when $mM \gtrsim 1$, especially near the horizon. Since the horizon radius obeys $r_h \sim M$ and the Compton wavelength associated to m is $\lambda \sim 1/m$, the rough inequality can be translated into $r_h/\lambda \gtrsim 1$. This could have been expected on physical grounds. On one hand, particles with much longer wavelengths (lower mass) are outside the validity of the approximation and it cannot give good results. On the other hand, since vacuum polarization happens most intensely near the horizon, then particles with wavelengths on the order of the horizon radius or less ($r_h/\lambda \gtrsim 1$) are well described by the approximation because they fit within the most probable characteristic geometric length of the fully quantum processes in a neighborhood of the horizon.

$d = 5$:

In five dimensions the metric function $f(r)$ for an asymptotically flat Reissner-Nordström black hole is

$$f(r) = 1 - \frac{2M}{r^2} + \frac{Q^2}{r^4}, \quad (61)$$

from which we find

$$[a_2] = \frac{1}{30r^{12}}[48M^2r^4 + Q^4(-17 + 460\xi + 60\xi^2) + 24Q^2r^2(M - 30M\xi + 2r^2(-1 + 5\xi))]. \quad (62)$$

Notice that, unlike in four dimensions, in five dimensions $[a_2]$ depends on the coupling constant ξ . From Eq. (61) it is straightforward to locate the outer horizon at $r_h^2 = M(1 + \sqrt{1 - (Q/M)^2})$. Letting $s = r^2 - r_h^2$, the near horizon behavior of $\langle\phi^2(x)\rangle_{d=5}$ is plotted in Figs. 2(a) and 3(a) for $\xi = 0$ and $\xi = 3/16$ (conformal coupling) respectively.

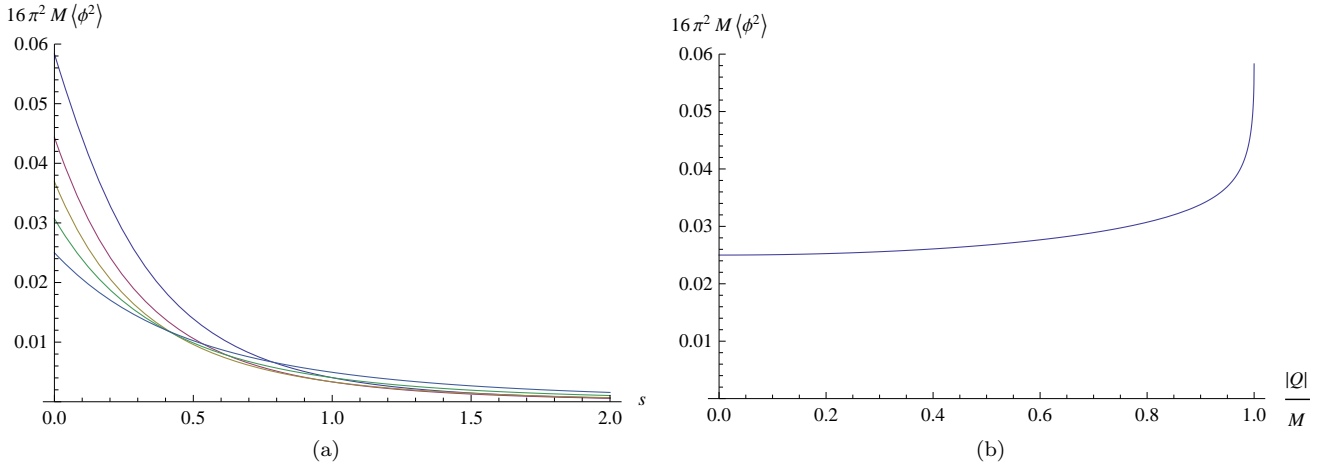


FIG. 2: Plot of $\langle \phi^2(s) \rangle$ for a minimally coupled field ($\xi = 0$) in a five-dimensional Reissner-Nordström black hole spacetime with $mM = 2$. (a) Near the horizon. From top to bottom at $s = 0$ the curves correspond to the cases $|Q|/M = 1.0, 0.99, 0.95, 0.8, 0.0$. The radial coordinate $s = r^2/M - 1 - \sqrt{1 - (Q/M)^2}$ is zero at the horizon, (b) On the horizon as a function of $|Q|/M$.

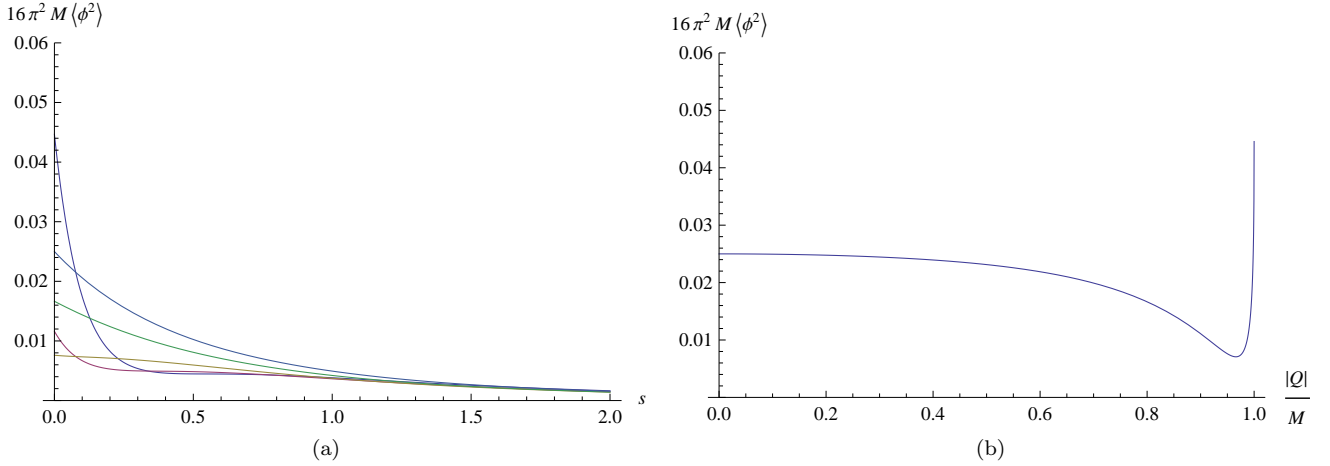


FIG. 3: Plot of $\langle \phi^2(s) \rangle$ for a conformally coupled field ($\xi = 3/16$) in a five-dimensional Reissner-Nordström black hole spacetime with $mM = 2$. (a) Near the horizon. From top to bottom at $s = 0$ the curves correspond to the cases $|Q|/M = 1.0, 0.0, 0.8, 0.99, 0.95$. The radial coordinate $s = r^2/M - 1 - \sqrt{1 - (Q/M)^2}$ is zero at the horizon, (b) On the horizon as a function of $|Q|/M$.

The behavior of the conformally coupled field is quite interesting and different from the behavior in four dimensions. The value of $\langle \phi^2(x) \rangle$ on the horizon is plotted in Figs. 2(b) and 3(b) for $\xi = 0$ and $\xi = 3/16$, respectively. The main feature is that while $\langle \phi^2(x) \rangle|_{x=x_h}$, the value on the horizon, increases monotonically with $|Q|/M$ for the minimally coupled field as for the four-dimensional case of Fig. 1(b), for the conformally coupled field there is a minimum value which occurs for a subextremal black hole with $|Q|/M \approx .97$. These results show that $\langle \phi^2(x) \rangle$ is well behaved near the horizon as might be expected, but some interesting variation arises for a non-minimally coupled field.

V. CONCLUSIONS

In this paper we have derived a compact expression for the DeWitt-Schwinger renormalization terms in d even dimensions. Beginning with the general d -dimensional formula for the DeWitt-Schwinger expansion, the divergent terms in the coincidence limit were isolated by considering an expansion in σ . From the properties of modified Bessel functions a useful integral representation for $K_\nu(z)$ in even dimensions was found in the coincidence limit. This integral may be used for the case of a scalar field at zero temperature without further modification. For a scalar field

at nonzero temperature T the Plana sum formula was used to convert the integral into a sum plus residual terms. The resulting formulas, Eq. (46) for a scalar field at zero temperature, and Eq. (47) for a scalar field at nonzero temperature T , are given in a form compatible with calculations of $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ in static spacetimes. These formulas will be particularly useful for calculating $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ in arbitrary black hole spacetimes of even dimension. The formulas found reproduce directly in the case $d = 4$ the results obtained by Anderson [40]. As a further example, the renormalization terms were calculated for six dimensions. Christensen remarked that calculating quantities such as $\langle T_{\mu\nu}(x)\rangle$ in dimensions greater than four “would be extremely long and would probably have to be done on a computer” [13] mainly due to the complexity of the renormalization problem. While the derivation of Eqs. (46) and Eq. (47) did not require any special computing power, it is certainly true that calculating quantities such as $[a_2]$ and $C_d^{\mathfrak{R}}$ for a particular spacetime would be quite lengthy and extremely prone to error without the aid of a computer. Lastly, the finite terms of the DeWitt-Schwinger expansion that are nonvanishing in the coincidence limit may be used as an approximation to $\langle\phi^2(x)\rangle$. It is shown that this reduces to a sum over the DeWitt coefficients, and is discussed in more detail in four and five dimensions.

As we have emphasized, in classical general relativity Einstein’s equations relate the spacetime curvature to the distribution of classical matter as encoded in the stress-energy tensor. Unfortunately the Universe is not so simple, being composed of quantum, rather than classical, matter. While some argue this indicates the need for a complete quantum theory of gravity, a first step, used here, is semiclassical general relativity, where the stress tensor appearing in Einstein’s equation is replaced by the expectation value of the stress tensor of quantum fields. Despite this objection, semiclassical general relativity has provided us with some impressive results and deep insight into the behavior of the Universe, but calculating the renormalized expectation values of $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ is quite difficult in curved spacetimes. We have dealt with this difficulty here for d -dimensional static spherical symmetric spacetimes in even dimensions. However, it is clear that, as it stands, the semiclassical theory is inadequate as a complete theory of gravity at the quantum level. By using the expectation value of the quantum fields, information about fluctuations of the fields, a defining characteristic of quantum field theory, is lost. At the very minimum, the semiclassical theory must be extended to incorporate some notion of fluctuations, and work in this area is being done by several authors; see for example [68, 69, 70, 71] and references therein.

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APPENDIX A: GENERALIZATION OF HOWARD’S IDENTITIES

For ε^{-2} , Howard [25] proved that

$$\frac{1}{\varepsilon^2} = - \sum_{n=1}^{\infty} \kappa^2 n \cos(n\kappa\varepsilon) - \frac{\kappa^2}{12} \quad (\text{A1})$$

for small, nonzero ε . Using Howard’s procedure for p an even integer this result is easily generalized, for the problem at hand, to

$$(m\varepsilon\sqrt{f})^{-p} = \left(\frac{i\kappa}{m\sqrt{f}}\right)^p \frac{1}{p!} \left[p \sum_{n=1}^{\infty} n^{p-1} \cos(n\kappa\varepsilon) + B_p \right], \quad (\text{A2})$$

where B_p are the Bernoulli numbers. For p an odd integer, the identity is generalized to

$$(m\varepsilon\sqrt{f})^{-p} = -i \left(\frac{i\kappa}{m\sqrt{f}}\right)^p \frac{p}{p!} \sum_{n=1}^{\infty} n^{p-1} \sin(n\kappa\varepsilon). \quad (\text{A3})$$

An integral representation was proved by Anderson, et. al. [42] by noting that

$$\int_{\lambda}^{\infty} dt \frac{\cos(\varepsilon t)}{t} = -\text{ci}(\lambda\varepsilon) \sim -(\ln(\lambda\varepsilon) + \gamma) \quad \text{as } \varepsilon \rightarrow 0. \quad (\text{A4})$$

By repeatedly taking the derivative of both sides of Eq. (A4) and then letting $\lambda \rightarrow 0$, one finds

$$\frac{1}{\varepsilon^{2n}} = \frac{(-1)^n}{\Gamma(2n)} \int_0^\infty dt t^{2n-1} \cos(\varepsilon t) \quad (\text{A5})$$

for even powers of ε , and

$$\frac{1}{\varepsilon^{2n-1}} = \frac{(-1)^{n+1}}{\Gamma(2n+1)} \int_0^\infty dt t^{2n} \sin(\varepsilon t) \quad (\text{A6})$$

for odd powers of ε . In Appendix B we must evaluate

$$\int_1^\infty dt t^{2n-1} \cos(\varepsilon t). \quad (\text{A7})$$

This is done by writing

$$\int_1^\infty dt t^{2n-1} \cos(\varepsilon t) = \int_0^\infty dt t^{2n-1} \cos(\varepsilon t) - \int_0^1 dt t^{2n-1} \cos(\varepsilon t) = \frac{(-1)^n \Gamma(2n)}{\varepsilon^{2n}} - \frac{1}{2n}, \quad (\text{A8})$$

where the $1/2n$ comes from expanding the solution of the second integral near $\varepsilon = 0$.

APPENDIX B: DERIVATION OF EQ. (39)

To obtain formula (39), begin with the recursion relation for the modified Bessel function,

$$K_{\nu+1}(z) = \frac{\nu}{z} K_\nu(z) - K'_\nu(z), \quad (\text{B1})$$

and the integral representation for $K_0(z)$ [63, Eq. (9.6.21)]

$$K_0(z) = \int_0^\infty \frac{\cos(zt) dt}{(t^2 + 1)^{1/2}}. \quad (\text{B2})$$

From the recursion relation $K_1(z) = -K'_0(z)$. Taking the derivative of Eq. (B2) and integrating once by parts gives

$$K_1(z) = \int_0^\infty \frac{t \sin(zt) dt}{(t^2 + 1)^{1/2}} = \lim_{s \rightarrow \infty} \sin(zt)(t^2 + 1)^{1/2} \Big|_0^s - z \int_0^\infty \cos(zt)(t^2 + 1)^{1/2} dt. \quad (\text{B3})$$

Taking the limit $z \rightarrow 0$, the first term vanishes and only the second term remains. Applying the recursion relation again to find $K_2(z)$, a fortuitous cancellation and another integration by parts results in

$$K_2(z) = \frac{z^2}{3} \int_0^\infty \cos(zt)(t^2 + 1)^{3/2} dt \approx \frac{(-1)^2 \sqrt{\pi}}{\Gamma(2 + \frac{1}{2})} \left(\frac{z}{2}\right)^2 \int_0^\infty \cos(zt)(t^2 + 1)^{2-1/2} dt. \quad (\text{B4})$$

By induction one is lead to

$$K_\nu(z) = \frac{(-1)^\nu \sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^2 \int_0^\infty \cos(zt)(t^2 + 1)^{2-1/2} dt. \quad (\text{B5})$$

One might worry that it is unreasonable to assume that the first term on the right hand side of Eq. (B3) vanishes, as the limit $z \rightarrow 0$ should only be taken at the end of the calculation, in which case the additional z^{-1} multiplying $K_1(z)$ leaves us with a problematic $\lim_{s \rightarrow \infty} z^{-1} \sin(zt)(t^2 + 1)^{1/2} \Big|_0^s$.

In fact, we can check that this integral representation reproduces the correct limiting behavior of $z^{-\nu} K_\nu(z)$ when z goes linearly to 0. First let

$$\int_0^\infty dt \cos(zt)(t^2 + 1)^{\nu-1/2} = \int_0^1 dt \cos(zt)(t^2 + 1)^{\nu-1/2} + \int_1^\infty dt \cos(zt)(t^2 + 1)^{\nu-1/2}. \quad (\text{B6})$$

For z in the coincidence limit we may expand the first integrand, $\cos(zt)(t^2 + 1)^{\nu-1/2} \approx (t^2 + 1)^{\nu-1/2}$, and solve the first integral

$$\int_0^\infty dt \cos(zt)(t^2 + 1)^{\nu-1/2} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \nu, \frac{3}{2}, -1\right) + \int_1^\infty dt \cos(zt)(t^2 + 1)^{\nu-1/2}. \quad (\text{B7})$$

For the second integral, expanding the integrand for $t > 1$ gives

$$\int_1^\infty dt \cos(zt)(t^2 + 1)^{\nu-1/2} = \Gamma\left(\nu + \frac{1}{2}\right) \sum_{n=1}^\infty \frac{1}{\Gamma(n)\Gamma\left(\nu - n + \frac{3}{2}\right)} \int_1^\infty dt \cos(zt)t^{2\nu-2n+1}. \quad (\text{B8})$$

This expression may be analyzed with the use of some identities proved by Anderson, et. al. [42], based on identities proved by Howard [25], and discussed further in Appendix A. It follows from the results of Appendix A that

$$\begin{aligned} \int_1^\infty dt \cos(zt)(t^2 + 1)^{\nu-1/2} = \Gamma\left(\nu + \frac{1}{2}\right) \left\{ \sum_{n=1}^\nu \frac{1}{\Gamma(n)\Gamma\left(\nu - n + \frac{3}{2}\right)} \left[\frac{(-1)^{\nu-n+1}\Gamma(2\nu - 2n + 2)}{z^{2\nu-2n+2}} - \frac{1}{2(\nu - n + 1)} \right] \right. \\ \left. - \frac{1}{\Gamma(\nu + 1)\sqrt{\pi}}(\ln z + \gamma) + \sum_{\nu+2}^\infty \frac{1}{\Gamma(n)\Gamma\left(\nu - n + \frac{3}{2}\right)} \int_1^\infty dt \cos(zt)t^{2\nu-2n+1} \right\}. \quad (\text{B9}) \end{aligned}$$

Renaming indices on the last sum, rearranging terms slightly and combining with the first part of Eq. (B7),

$$\begin{aligned} \int_0^\infty dt \cos(zt)(t^2 + 1)^{\nu-1/2} = \Gamma\left(\nu + \frac{1}{2}\right) \left\{ \sum_{n=1}^\nu \frac{(-1)^{\nu-n+1}\Gamma(2\nu - 2n + 2)}{\Gamma(n)\Gamma\left(\nu - n + \frac{3}{2}\right)z^{2\nu-2n+2}} - \frac{1}{\Gamma(\nu + 1)\sqrt{\pi}} \left[\ln z + \gamma \right. \right. \\ \left. \left. + \sum_{n=1}^\nu \frac{\Gamma(\nu + 1)\sqrt{\pi}}{2(\nu - n + 1)\Gamma(n)\Gamma\left(\nu - n + \frac{3}{2}\right)} - \sum_{n=1}^\infty \frac{\Gamma(\nu + 1)\sqrt{\pi}}{2n\Gamma(\nu + n + 1)\Gamma\left(\frac{1}{2} - n\right)} - \frac{\Gamma(\nu + 1)\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \nu, \frac{3}{2}, -1\right) \right] \right\}. \quad (\text{B10}) \end{aligned}$$

The last sum may also be expressed in terms of a hypergeometric function, and the limiting behavior is finally

$$\begin{aligned} z^{-\nu} K_\nu(z) = \sum_{n=1}^\nu \frac{(-1)^{2\nu-n+1}\Gamma(2\nu - 2n + 2)\sqrt{\pi}}{2^\nu \Gamma(n)\Gamma\left(\nu - n + \frac{3}{2}\right)z^{2\nu-2n+2}} - \frac{(-1)^\nu}{2^\nu \Gamma(\nu + 1)} \left[\ln z + \gamma + \sum_{n=1}^\nu \frac{\Gamma(\nu + 1)\sqrt{\pi}}{2(\nu - n + 1)\Gamma(n)\Gamma\left(\nu - n + \frac{3}{2}\right)} \right. \\ \left. + \frac{\Gamma(\nu + 1)}{4(\Gamma(\nu + 2))} {}_3F_2\left(1, 1, \frac{3}{2}, 2, 2 + \nu, -1\right) - \frac{\Gamma(\nu + 1)\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \nu, \frac{3}{2}, -1\right) \right] + O(z). \quad (\text{B11}) \end{aligned}$$

This result does not look the same as the expansion given in Eq. (30), but we checked the expansion explicitly and believe the agreement is exact. To have exact agreement, the last three terms in brackets of Eq. (B11) must be identical to $-\sum_{n=1}^\nu n^{-1} - \ln 2$. The two expressions were evaluated numerically with a precision of 100 digits up to $\nu = 100$ and in each case were found to agree within the working precision. While this numerical evaluation does not constitute a rigorous proof, it is a strong indication that the two expressions agree exactly and therefore the integral representation is valid.

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